

## AN EXTENSION OF ERGODIC THEORY FOR GAUSS-TYPE MAPS

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ABSTRACT. The impetus to this work is the need to show that for positive reals  $\alpha$  and  $\beta$ , the functions

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \quad m, n \in \mathbb{Z}_+ \cup \{0\},$$

span a weak-star dense subspace of  $H_+^\infty(\mathbb{R})$  if and only if  $0 < \alpha\beta \leq 1$ . Here,  $H_+^\infty(\mathbb{R})$  is the subspace of  $L^\infty(\mathbb{R})$  which consists of those functions whose Poisson extensions to the upper half-plane are holomorphic. In earlier work in the  $L^\infty(\mathbb{R})$  context, we showed the relevance of the analysis of the dynamics of the Gauss-type mapping  $x \mapsto -\beta/x \bmod 2\mathbb{Z}$  for this problem (if  $\alpha = 1$ , which can be assumed by a scaling argument). For  $\beta = 1$ , the ergodic properties of the absolutely continuous invariant measure  $(1 - x^2)^{-1}dx$  on the interval  $I_1 = ]-1, 1[$  turned out to be crucial. In the present setting, although the norm in  $H_+^\infty(\mathbb{R})$  is the same as in  $L^\infty(\mathbb{R})$ , in the real sense, it is much finer. The corresponding real space is  $H_\otimes^\infty(\mathbb{R})$ , which consists of all the functions in  $L^\infty(\mathbb{R})$  whose modified Hilbert transform is also in  $L^\infty(\mathbb{R})$ . From the real perspective, our task is clearer: We need to show that the functions

$$e^{i\pi m t}, \quad e^{-i\pi\beta n/t}, \quad m, n \in \mathbb{Z},$$

span a weak-star dense subspace in  $H_\otimes^\infty(\mathbb{R})$  precisely when  $0 < \beta \leq 1$ . The predual of  $H_\otimes^\infty(\mathbb{R})$  is identified with a space  $\mathcal{V}(\mathbb{R})$  of distributions on  $\mathbb{R}$ , obtained as the sum of  $L^1(\mathbb{R})$  and  $\mathbf{H}L_0^1(\mathbb{R})$ , where  $L_0^1(\mathbb{R})$  is the codimension 1 subspace of  $L^1(\mathbb{R})$  of functions with integral 0. While the space  $\mathcal{V}(\mathbb{R})$  consists of distributions, it also can be said to consist of weak- $L^1$  functions, and a theorem of Kolmogorov guarantees that the viewpoints are equivalent. It is in a sense the extension of  $L^1(\mathbb{R})$  which is analogous to having  $\text{BMO}(\mathbb{R})$  as the extension of  $L^\infty(\mathbb{R})$ . Whereas transfer (and subtransfer) operators usually act on  $L^1$  of an interval (or, more generally, on the finite Borel measures on that interval), here we consider the corresponding operators acting on the restriction of  $\mathcal{V}(\mathbb{R})$  to the interval  $I$  in question, denoted  $\mathcal{V}(I)$ . In the convex body of invariant absolutely continuous probability measures an element is ergodic if it is an extreme point. In the setting of infinite ergodic theory, which is more relevant here, ergodicity means that no element of  $L^1$  on the interval can be invariant (under the transformation, or, which is the same, under the transfer operator). We study mainly a particular instance of infinite ergodic theory, and extend the concept of ergodicity by showing that for the transformation  $x \mapsto -\beta/x \bmod 2\mathbb{Z}$  on  $I_1$ , (i) for  $0 < \beta < 1$ , there is no nontrivial subtransfer operator invariant distribution in  $\mathcal{V}(I_1)$ , whereas (ii) for  $\beta = 1$ , there is no nontrivial transfer operator invariant *odd* distribution in  $\mathcal{V}(I_1)$ . The oddness helps in the proof, but we expect it to be superfluous. The conclusion is nevertheless strong enough to supply an affirmative answer to our original density problem. To obtain the results (i)-(ii), we develop new tools, which offer a novel amalgam of ideas from Ergodic Theory with ideas from Harmonic Analysis. We need to handle in a subtle way series of powers of transfer operators, a rather intractable problem where even the recent advances by Melbourne and Terhesiu do not apply. More specifically, our approach involves a splitting of the Hilbert kernel induced by the transfer operator. The careful analysis of this splitting involves detours to the Hurwitz zeta function as well as to the theory of totally positive matrices.

## 1. INTRODUCTION

2000 *Mathematics Subject Classification.* Primary 42B10, 42B20, 35L10, 42B37, 42A64. Secondary 37A45, 43A15.

*Key words and phrases.* Transfer operator, Hilbert transform, completeness, Klein-Gordon equation.

The research of Hedenmalm was supported by Vetenskapsrådet (VR). The research of both authors was supported by Plan Nacional ref. MTM2012-35107 and by Junta Andalucía P12-FQM-633, FQM-260.

**1.1. An elementary example: the doubling map of an interval.** Let us consider the doubling map of the unit interval  $I_1^+ := [0, 1]$ , given by  $\theta(t) := 2t \bmod \mathbb{Z}$ ; to be more precise, we put  $\theta(t) = 2t$  on  $[0, \frac{1}{2}]$ , and  $\theta(t) = 2t - 1$  on  $[\frac{1}{2}, 1]$ . For  $h \in L^\infty(I_1^+)$  and  $g \in L^1(I_1^+)$ , we have the identity

$$\int_0^1 h \circ \theta(t) g(t) dt = \int_0^1 h(t) \Theta g(t) dt,$$

where  $\Theta$  is the associated *transfer operator*

$$\Theta g(t) := \frac{1}{2} \left( g\left(\frac{t}{2}\right) + g\left(\frac{t+1}{2}\right) \right), \quad t \in I_1^+.$$

The function  $g \in L^1(I_1^+)$  (and the corresponding absolutely continuous measure  $g(t)dt$ ) is said to be *invariant* with respect to the doubling map  $\theta$  if  $\Theta g = g$ . We quickly check that the constant function  $g_0(t) \equiv 1$  is invariant, and wonder if there are any other invariant functions beyond the scalar multiples of  $g_0 = 1$ . To analyze the situation, Fourier analysis comes very handy. We expand the function  $g \in L^1(I_1^+)$  in a Fourier series

$$g(t) \sim \sum_{j=-\infty}^{+\infty} \hat{g}(j) e^{i2\pi jt}, \quad t \in I_1^+,$$

which actually need not converge pointwise, but this does not bother us. The Fourier series associated with  $\Theta g$  is then

$$\Theta g(t) \sim \frac{1}{2} \sum_{j=-\infty}^{+\infty} \hat{g}(j) (e^{i\pi jt} + e^{i\pi j(t+1)}) = \frac{1}{2} \sum_{j=-\infty}^{+\infty} \hat{g}(j) (1 + (-1)^j) e^{i\pi jt} = \sum_{k=-\infty}^{+\infty} \hat{g}(2k) e^{i2\pi kt},$$

and, by iteration,

$$\Theta^n g(t) \sim \sum_{k=-\infty}^{+\infty} \hat{g}(2^n k) e^{i2\pi kt}, \quad n = 0, 1, 2, \dots$$

If  $g$  solves the more general eigenvalue problem  $\Theta g = \lambda g$  for some complex nonzero scalar  $\lambda \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ , then we see by equating Fourier coefficients that we must have

$$(1.1.1) \quad \hat{g}(k) = \lambda^{-n} \hat{g}(2^n k), \quad k \in \mathbb{Z},$$

for  $n = 0, 1, 2, \dots$ . By plugging in  $k = 0$ , we derive from the above equation (1.1.1) that  $\lambda = 1$  is the only possibility, provided that  $\hat{g}(0) \neq 0$ . Moreover, for  $k \in \mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$ , we know from the Riemann-Lebesgue lemma that

$$\hat{g}(2^n k) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

which lets us to conclude from (1.1.1) that

$$\hat{g}(k) = 0, \quad k \in \mathbb{Z}^\times,$$

provided that  $|\lambda| \geq 1$ . In this case,  $g$  is of course constant, and if the constant is nonzero, then we also know that  $\lambda = 1$ . In particular, the only invariant functions in  $L^1(I_1^+)$  are the constants. This observation is an equivalent reformulation of the well-known ergodicity of the doubling map with respect to the uniform measure on the interval  $I_1^+$  (see below).

**OBSERVATION:** As we look back at the argument just presented, we realize that we did not use all that much about the function  $g$ , just that the conclusion of the Riemann-Lebesgue lemma holds. So in principle, we could replace  $g$  by a finite Borel measure, and obtain the same conclusion, if the Fourier coefficients of the measure tend to 0 at infinity. Such measures are known as *Rajchman measures*, and have been studied in depth in harmonic analysis. But the point of view we want to present here goes beyond that setting. *We are in fact at liberty to replace  $g$  by a distribution with a periodic extension*, so that it has a Fourier series expansion, and so long as its Fourier coefficients  $\hat{g}(j)$  tend to 0 as  $|j| \rightarrow +\infty$ , the argument works, and tells us that the constants are the unique  $\Theta$ -invariant elements of this much wider space of distributions. Such periodic distributions  $g$  which Fourier coefficients  $\hat{g}(j)$  which tend to 0 as  $|j| \rightarrow +\infty$  deserve to be called *Rajchman pseudomeasures* (cf. [13]). This uniqueness within the

Rajchman pseudomeasures can be understood as *an extension of standard ergodic theory for the doubling map with respect to the constant density 1*. Indeed, an easy argument shows that the following are equivalent, for an invariant probability measure  $\mu$ :

- (i)  $\mu$  is ergodic, and
- (ii) whenever  $\nu$  is a finite (signed) invariant measure, absolutely continuous with respect to  $\mu$ , then  $\nu$  is a scalar multiple of  $\mu$ .

This is probably well-known. For completeness, we supply the relevant argument. Note first that we may restrict to real measures and real scalars in (ii). The implication (i)  $\implies$  (ii) is pretty standard and runs as follows. By replacing  $\nu$  by the sum of  $\nu$  and an appropriate scalar multiple of  $\mu$ , we reduce to the case when  $\nu$  has signed mass 0. Then, unless  $\nu = 0$ , we split  $\nu$  into positive and negative parts, which are seen to be left invariant by the transfer operator, as otherwise the transfer operator applied to  $\nu$  would have smaller total variation than  $\nu$  itself. But then the support (or rather, carrier) sets for the positive and negative parts are necessarily invariant under the transformation, in violation of the ergodic assumption (i), and the only remaining alternative is that  $\nu = 0$ , i.e., the assertion (ii) holds. The remaining implication (ii)  $\implies$  (i) is even simpler. We prove the contrapositive implication, and assume that (i) fails, so that  $\mu$  is not ergodic. Then  $\mu$  is not an extreme point in the convex body of all invariant probability measures, and hence it splits as a nontrivial convex combination of two invariant probability measures. Both measures are assumed different than  $\mu$  itself, and each one is obviously absolutely continuous with respect to  $\mu$ , which shows that (ii) fails as well.

**1.2. The Gauss-type maps on the symmetric unit interval.** It was the fact that the doubling map is piecewise affine that made it amenable to methods from Fourier analysis. This is not the case for the Gauss-type map  $\tau_\beta$  acting on the symmetric interval  $I_1 := ]-1, 1[$ , defined in the following fashion. First, we let  $\{x\}_2$  denote the *even-fractional part of  $x$* , by which we mean the unique number in the half-open interval  $\tilde{I}_1 := ]-1, 1]$  with  $x - \{x\}_2 \in 2\mathbb{Z}$ . The Gauss-type map  $\tau_\beta : \tilde{I}_1 \rightarrow \tilde{I}_1$  is given by the expression

$$\tau_\beta(x) := \left\{ -\frac{\beta}{x} \right\}_2.$$

Here and in the sequel,  $\beta$  is assumed real with  $0 < \beta \leq 1$ . The basic properties of  $\tau_\beta$  are well-known, see, e.g. [12]. We outline the basic aspects below, which are mainly based on the work of Thaler [22] and Lin [14]. For  $0 < \beta < 1$ , the set  $I_1 \setminus \tilde{I}_\beta$  acts as an *attractor* for the iterates under  $\tau_\beta$ , and inside the attractor  $I_1 \setminus \tilde{I}_\beta$ , the orbits form 2-cycles. Here,  $\tilde{I}_\beta$  denotes the symmetric interval  $\tilde{I}_\beta := [-\beta, \beta]$ . For  $\beta = 1$ , on the other hand, we are in the setting of infinite ergodic theory, where  $(1 - x^2)^{-1}dx$  is the ergodic invariant measure. The reason is that the endpoint 1 (which for all essential purposes may be identified with the left-hand endpoint  $-1$  for the dynamics) is only weakly repelling. The *transfer operator*  $\mathcal{T}_\beta$  linked with the map  $\tau_\beta$  is the operator which can be understood as taking the unit point mass  $\delta_x$  at a point  $x \in \tilde{I}_1$  to the unit point mass  $\delta_{\tau_\beta(x)}$  at the point  $\tau_\beta(x)$ . To be more definitive, for a function  $f \in L^1(I_1)$ , we write  $f$  as an integral of point masses,

$$(1.2.1) \quad f(x) = \int_{I_1} f(t) \delta_x(t) dt = \int_{I_1} f(t) \delta_t(x) dt,$$

understood in the sense of distribution theory, and say that

$$(1.2.2) \quad \mathcal{T}_\beta f(x) := \int_{I_1} f(t) \mathcal{T}_\beta \delta_t(x) dt = \int_{I_1} f(t) \delta_{\tau_\beta(t)}(x) dt, \quad x \in I_1,$$

which is seen to be the same as the more explicit formula

$$(1.2.3) \quad \mathcal{T}_\beta f(x) = \begin{cases} \sum_{j \in \mathbb{Z}} \frac{\beta}{(x + 2j)^2} f\left(-\frac{\beta}{x + 2j}\right), & x \in I_1, \\ 0, & x \in \mathbb{R} \setminus I_1, \end{cases}$$

which has the added advantage that the values off the interval  $I_1$  are declared to vanish. The behavior of  $\tau_\beta$  is rather uninteresting on the attractor  $I_1 \setminus \bar{I}_\beta$ , and for this reason, we introduce the *subtransfer operator*  $\mathbf{T}_\beta$  which discards the point masses from the attractor. In other words, we put

$$(1.2.4) \quad \mathbf{T}_\beta f(x) := \mathcal{T}(1_{\bar{I}_\beta} f)(x) = \int_{\bar{I}_\beta} f(t) \mathcal{T}_\beta \delta_t(x) dt = \int_{\bar{I}_\beta} f(t) \delta_{\tau_\beta(t)}(x) dt, \quad x \in I_1.$$

In more direct terms, this is the same as

$$(1.2.5) \quad \mathbf{T}_\beta f(x) := \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(2j+x)^2} f\left(-\frac{\beta}{2j+x}\right), \quad x \in I_1,$$

which we see from (1.2.3). Here,  $\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$ , as before. For  $0 < \beta < 1$ , the  $\tau_\beta$ -orbit of a point  $x \in I_1$  falls into the attractor  $I_1 \setminus \bar{I}_\beta$  almost surely. In terms of the subtransfer operator  $\mathbf{T}_\beta$ , this means that

$$(1.2.6) \quad \forall f \in L^1(I_1) : \quad \mathbf{T}_\beta^n f \rightarrow 0 \quad \text{in } L^1(I_1), \quad \text{if } 0 < \beta < 1.$$

For  $\beta = 1$ , things are a little more subtle. Nevertheless, it can be shown that

$$(1.2.7) \quad \forall f \in L^1(I_1) : \quad 1_{I_\eta} \mathbf{T}_1^n f \rightarrow 0 \quad \text{in } L^1(I_1),$$

for every fixed real  $\eta$  with  $0 < \eta < 1$ . Here, as expected,  $I_\eta$  is the symmetric interval  $I_\eta := ]-\eta, \eta[$ . In particular, there is no nontrivial function  $f \in L^1(I_1)$  with  $\mathbf{T}_\beta f = \lambda f$  for any  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq 1$  and any  $\beta$  with  $0 < \beta \leq 1$ .

In [12], the subtransfer operator  $\mathbf{T}_\beta$  was shown to extend to a bounded operator on the space  $\mathfrak{L}(I_1)$ , whose elements are distributions on  $I_1$ . The space  $\mathfrak{L}(I_1)$  consists of the restrictions to the open interval  $I_1$  of the distributions in the space

$$\mathfrak{L}(\mathbb{R}) := L^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R}),$$

supplied with the induced quotient norm, as we mod out with respect to all the distributions whose support is contained in  $\mathbb{R} \setminus I_1$ . The quotient norm comes from the norm on the space  $\mathfrak{L}(\mathbb{R})$ , which is given by

$$(1.2.8) \quad \|u\|_{\mathfrak{L}(\mathbb{R})} := \inf \left\{ \|f\|_{L^1(\mathbb{R})} + \|g\|_{L^1(\mathbb{R})} : u = f + \mathbf{H}g, f \in L^1(\mathbb{R}), g \in L_0^1(\mathbb{R}) \right\},$$

and we should mention that the  $\mathfrak{L}(\mathbb{R})$  is in the natural sense *the predual of the real  $H^\infty$ -space on the line*, denoted by  $H_\otimes^\infty(\mathbb{R})$ , which consists of all the functions in  $L^\infty(\mathbb{R})$  whose modified Hilbert transform also is in  $L^\infty(\mathbb{R})$ . In the definition of the space  $\mathfrak{L}(\mathbb{R})$ , the letter  $\mathbf{H}$  stands for the *Hilbert transform*, given by the principal value integral

$$\mathbf{H}g(x) := \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} g(t) \frac{dt}{x-t} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{R} \setminus [x-\epsilon, x+\epsilon]} g(t) \frac{dt}{x-t},$$

and  $L_0^1(\mathbb{R})$  is the codimension 1 subspace

$$L_0^1(\mathbb{R}) := \left\{ g \in L^1(\mathbb{R}) : \int_{\mathbb{R}} g(t) dt = 0 \right\}.$$

By a theorem of Kolmogorov, the Hilbert transform of an  $L^1(\mathbb{R})$  is well-defined pointwise almost everywhere as a function in the quasi-Banach space  $L^{1,\infty}(\mathbb{R})$  of weak- $L^1$  functions. More generally, if  $E \subset \mathbb{R}$  is Lebesgue measurable with positive length, the *weak- $L^1$  space*  $L^{1,\infty}(E)$  consists of all measurable functions  $f : E \rightarrow \mathbb{C}$  with finite quasinorm

$$(1.2.9) \quad \|f\|_{L^{1,\infty}(E)} := \sup \left\{ \lambda |N_f(\lambda)| : \lambda > 0 \right\},$$

where  $N_f(\lambda)$  denotes the set

$$N_f(\lambda) := \{t \in E : |f(t)| > \lambda\},$$

and the absolute value sign assigns the linear length to given set. Kolmogorov's theorem allows us to think of the distributions (or pseudomeasures) in  $\mathfrak{L}(\mathbb{R})$  as elements of  $L^{1,\infty}(\mathbb{R})$ , so that in particular,  $\mathfrak{L}(I_1)$  can be identified with a subspace of  $L^{1,\infty}(I_1)$ , the corresponding weak- $L^1$

space on the interval  $I_1$ . For the pointwise interpretation, the formula (1.2.5) for the operator  $T_\beta$  remains valid. We will work mainly in the setting of distribution theory. When we need to speak of the pointwise function rather than the distribution  $u$ , we write  $\text{vp}(u)$  in place of  $u$ , and call it the *valeur au point*. So “vp” maps from distributions to functions.

On the space  $L^1(I_1)$ , the subtransfer operators  $T_\beta$  all act contractively. This is not the case with the extension to  $\mathfrak{L}(I_1)$ .

**Theorem 1.2.1.** *Fix  $0 < \beta \leq 1$ . Then the operator  $T_\beta : \mathfrak{L}(I_1) \rightarrow \mathfrak{L}(I_1)$  is bounded, but its norm exceeds 1.*

The proof of Theorem 1.2.1 is supplied in Subsection 9.7.

A decomposition analogous to (1.2.1) holds for distributions  $u \in \mathfrak{L}(I_1)$  as well, only we would need two integrals, one with  $\delta_t(x)$  and the other with  $H\delta_t(x)$  (and the latter integral should be taken over a bigger interval, e.g.  $I_2 = ]-2, 2[$  to allow for tails). Thinking physically, we allow for two kinds of “particles”, focused particles  $\delta_t$  as well as spread-out particles  $H\delta_t$ . Then  $\mathfrak{L}(I_1)$  is a kind of state space, and  $T_\beta$  acts on this state space. It is then natural to ask whether there is a nontrivial invariant state under  $T_\beta$ . More generally, we would ask whether there exists a  $u \in \mathfrak{L}(I_1)$  with  $T_\beta u = \lambda u$  for any scalar  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq 1$ . To appreciate the subtlety of this question, we note that in the slightly larger space  $L^{1,\infty}(I_1)$ , there are plenty of invariant states  $u \in L^{1,\infty}(I_1)$  with  $T_\beta u = u$ , see the example provided in Remark 11.2.1. That example is constructed as the Hilbert transform of the difference of two Dirac point masses, with one point inside  $I_1$  and the other point outside  $I_1$ . The example in fact suggests that within the space of Hilbert transforms of finite Borel measures, the invariant states might possess an intricate and interesting structure. In the space  $\mathfrak{L}(\mathbb{R})$ , which contains the Hilbert transforms of the absolutely continuous measures, this is however not the case.

**Theorem 1.2.2.** *Fix  $0 < \beta < 1$ . For  $u_0 \in \mathfrak{L}(I_1)$ , we have the asymptotic decay  $\text{vp}(T_\beta^N u_0) \rightarrow 0$  in  $L^{1,\infty}(I_1)$  as  $N \rightarrow +\infty$ .*

So, although  $T_\beta$  has norm that exceeds 1 on  $\mathfrak{L}(I_1)$ , the orbit of a given  $u \in \mathfrak{L}(I_1)$  converges to 0 in the weaker sense of the quasinorm in  $L^{1,\infty}(I_1)$ . In other words, the  $L^{1,\infty}$ -quasinorm serves as a *Lyapunov energy* for the asymptotic stability of the  $T_\beta$ -orbits. In the setting of the smaller space  $L^1(I_1)$ , this convergence amounts to the statement that the basin of attraction of the attractor  $I_1 \setminus \bar{I}_\beta$  contains almost every point of the interval  $I_1$ . Apparently, this property extends to the larger space  $\mathfrak{L}(I_1)$ , but not to e.g.  $L^{1,\infty}(I_1)$  (see Remark 11.2.1). The proof of Theorem 1.2.2 is supplied in Subsection 11.2.

**Corollary 1.2.3.** *Fix  $0 < \beta < 1$ . If  $T_\beta u = \lambda u$  for some  $u \in \mathfrak{L}(I_1)$  and some scalar  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq 1$ , then  $u = 0$ .*

In other words, for  $0 < \beta$ , the point spectrum of the operator  $T_\beta : \mathfrak{L}(I_1) \rightarrow \mathfrak{L}(I_1)$  is contained in the open unit disk  $\mathbb{D}$ . It is clear that Corollary 1.2.3 follows from Theorem 1.2.2.

Our understanding is slightly less complete for  $\beta = 1$ . We recall that a distribution, defined on a symmetric interval about 0, is *odd* if its action on the even test functions equals 0.

**Theorem 1.2.4.** ( $\beta = 1$ ) *For odd  $u_0 \in \mathfrak{L}(I_1)$ , we have the asymptotic decay  $1_{I_\eta} \text{vp}(T_1^N u_0) \rightarrow 0$  in  $L^{1,\infty}(I_1)$  as  $N \rightarrow +\infty$  for each  $\eta$  with  $0 < \eta < 1$ .*

The proof, which is supplied in Subsection 14.2, is *much much more sophisticated* than that of Theorem 1.2.2. It uses the full strength of the machinery developed around a subtle *dynamical decomposition* of the odd part of the Hilbert kernel. We believe that a similar dynamical decomposition is available for the even part of the Hilbert kernel as well, which would remove the need for the oddness assumption. Again, the  $L^{1,\infty}(I_\eta)$ -quasinorms serve as Lyapunov energy functionals, for each  $\eta$  with  $0 < \eta < 1$ . In the setting of the smaller space  $L^1(I_1)$ , the corresponding statement is based on the fact that the dynamics of  $\tau_1$  has  $\pm 1$  as a weakly repelling fixed point, so that the ergodic invariant measure for  $L^1(I_1)$  get infinite mass and

cannot be in  $L^1(I_1)$ . It follows immediately from Theorem 1.2.4 that the point spectrum of the operator  $T_1 : \mathfrak{L}_{\text{odd}}(I_1) \rightarrow \mathfrak{L}_{\text{odd}}(I_1)$  is contained in the open unit disk  $\mathbb{D}$ . In particular, there is no  $T_1$ -invariant element of  $\mathfrak{L}_{\text{odd}}(I_1)$ , the subspace of the odd distributions in  $\mathfrak{L}(I_1)$ .

**Corollary 1.2.5.** ( $\beta = 1$ ) *If  $T_1 u = \lambda u$  for some odd  $u \in \mathfrak{L}(I_1)$  and some scalar  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq 1$ , then  $u = 0$ .*

As already mentioned, this corollary is an immediate consequence of Theorem 1.2.4.

From a dynamical perspective, it is quite natural to introduce the odd-even symmetry, as the transformation  $\tau_\beta$  itself is odd:  $\tau_\beta(-x) = -\tau_\beta(x)$  (except possibly at the endpoints  $\pm 1$ ). E.g., in connection with the partial fraction expansions with even partial quotients, it is standard to keep track of only the orbit of the absolute values on the interval  $I_1^+$ . Note that clearly, the subtransfer operators  $T_\beta$  preserve odd-even symmetry. As for the remaining even symmetry case, we observe that  $\mathbf{H}(\delta_{-1} - \delta_1) = \frac{2}{\pi} \text{pv}(1 - x^2)^{-1}$  which is even and equals (a constant multiple of) the density of the ergodic invariant measure.

*Remark 1.2.6.* In view of the Observation in Subsection 1.1, Corollaries 1.2.3 and 1.2.5 go beyond the standard notion of ergodicity. The main point is that we insert *distribution theory in place of measure theory*. We have not been able to find any appropriate references for this in the literature, but suggest some relevance of the works [2], [3] for the discrete setting, and [4] for flows.

**1.3. Applications to the problem of completeness of a system of unimodular functions.** As an application to Corollaries 1.2.3 and 1.2.5, we have the following result on the completeness of the nonnegative integer powers of two singular inner functions in the weak-star topology of the space  $H_+^\infty(\mathbb{R})$  of functions which extend boundedly and holomorphically to the upper half-plane.

**Theorem 1.3.1.** *Fix two positive reals  $\alpha, \beta$ . Then the functions*

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \quad m, n = 0, 1, 2, \dots,$$

*which are elements of  $H_+^\infty(\mathbb{R})$ , span together a weak-star dense subspace of  $H_+^\infty(\mathbb{R})$  if and only if  $\alpha\beta \leq 1$ .*

Note that the “only if” part of Theorem 1.3.1, is quite simple, as for instance the work in [5] shows that in case  $\alpha\beta > 1$ , the weak-star closure of the linear span in question has infinite codimension in  $H_+^\infty(\mathbb{R})$ . Hence the main thrust of the theorem is the “if” part. The proof of Theorem 1.3.1 is supplied in two installments: for  $\alpha\beta < 1$  in Subsection 11.1, and for  $\alpha\beta = 1$  in Subsection 14.1.

A standard Möbius mapping brings the upper half-plane to the unit disk  $\mathbb{D}$ , and identifies the space  $H_+^\infty(\mathbb{R})$  with  $H^\infty(\mathbb{D})$ , the space of all bounded holomorphic functions on  $\mathbb{D}$ . For this reason, Theorem 1.3.1 is equivalent to the following assertion, which we state as a corollary.

**Corollary 1.3.2.** *Fix two positive reals  $\lambda_1, \lambda_2$ . Then the linear span of the functions*

$$\phi_1(z)^m = \exp\left(m\lambda_1 \frac{z+1}{z-1}\right) \quad \text{and} \quad \phi_2(z)^n = \exp\left(n\lambda_2 \frac{z-1}{z+1}\right), \quad m, n = 0, 1, 2, \dots,$$

*is weak-star dense in  $H^\infty(\mathbb{D})$  if and only if  $\lambda_1\lambda_2 \leq \pi^2$ .*

We suppress the trivial proof of the corollary.

*Remark 1.3.3.* Clearly, Theorem 1.3.1 supplies a complete and affirmative answer to Problems 1 and 2 in [15]. We recall the question from [15]: the issue was raised whether the algebra generated by the two inner functions

$$\phi_1(z) = \exp\left(\lambda_1 \frac{z+1}{z-1}\right) \quad \text{and} \quad \psi_2(z) = \exp\left(\lambda_2 \frac{z-1}{z+1}\right)$$

for  $0 < \lambda_1, \lambda_2 < +\infty$ , is weak-star dense in  $H^\infty(\mathbb{D})$  if and only if  $\lambda_1 \lambda_2 \leq \pi^2$ . The “only if” was understood already in [15], while it is a consequence of Theorem 1.3.1 that if  $\lambda_1 \lambda_2 \leq \pi^2$ , then the linear span of the functions

$$\phi_1(z)^m = \exp\left(m\lambda_1 \frac{z+1}{z-1}\right) \quad \text{and} \quad \psi_2(z)^n = \exp\left(n\lambda_2 \frac{z-1}{z+1}\right), \quad m, n = 0, 1, 2, \dots,$$

is weak-star dense set in  $H^\infty(\mathbb{D})$ , without the need to resort to the whole algebra.

The  $L^\infty(\mathbb{R})$  analogue of Theorem 1.3.1 was obtained in [11]. In the context of Theorem 1.3.1, the  $L^\infty(\mathbb{R})$  result leads to completeness in the weak-star topology of  $\text{BMOA}_+(\mathbb{R})$ , the BMOA space of the upper half-plane. The latter assertion is substantially weaker than Theorem 1.3.1, as it is not difficult to exhibit a sequence of functions in  $H_+^\infty(\mathbb{R})$  which fails to be weak-star complete in  $H_+^\infty(\mathbb{R})$ , but is weak-star complete in  $\text{BMOA}_+(\mathbb{R})$ .

## 2. BASIC PROPERTIES OF THE DYNAMICS OF GAUSS-TYPE MAPS ON INTERVALS

**2.1. Notation for intervals.** For a positive real  $\gamma$ , let  $I_\gamma := ]-\gamma, \gamma[$  denote the corresponding symmetric open interval, and let  $I_\gamma^+ := ]0, \gamma[$  be the positive side of the interval  $I_\gamma$ . At times, we will need the half-open intervals  $\tilde{I}_\gamma := ]-\gamma, \gamma]$  and  $\tilde{I}_\gamma^+ := [0, \gamma[$ , as well as the closed intervals  $\bar{I}_\gamma := [-\gamma, \gamma]$  and  $\bar{I}_\gamma^+ := [0, \gamma]$ .

**2.2. Dual action notation.** For a Lebesgue measurable subset  $E$  of the real line  $\mathbb{R}$ , we write

$$\langle f, g \rangle_E := \int_E f(t)g(t)dt,$$

whenever  $fg \in L^1(E)$ . This will be of interest mainly when  $E$  is an open interval, and in this case, we use the same notation to describe the dual action of a distribution on a test function. For a set  $E \subset \mathbb{R}$ ,  $1_E$  stands for the characteristic function of  $E$ , which equals 1 on  $E$  and vanishes elsewhere. So, in particular, we see that

$$\langle f, g \rangle_E = \langle 1_E f, g \rangle_{\mathbb{R}} = \langle 1_E f, 1_E g \rangle_{\mathbb{R}}.$$

**2.3. Gauss-type maps on intervals.** For background material in Ergodic Theory, we refer to e.g. the book [6].

For  $N = 2, 3, 4, \dots$ , the  $N$ -step wandering subset is given by

$$(2.3.1) \quad \mathcal{E}_{\beta, N} := \left\{ x \in \bar{I}_\beta : \tau_\beta^n(x) \in \bar{I}_\beta \text{ for } n = 1, \dots, N-1 \right\},$$

where  $\tau_\beta^n := \tau_\beta \circ \dots \circ \tau_\beta$  ( $n$ -fold composition). We also agree that  $\mathcal{E}_{\beta, 1} := \bar{I}_\beta$ . The sets  $\mathcal{E}_{\beta, N}$  get smaller as  $N$  increases, and we form their intersections

$$(2.3.2) \quad \mathcal{E}_{\beta, \infty} := \bigcap_{N=1}^{+\infty} \mathcal{E}_{\beta, N},$$

The cone of positive functions consists of all integrable functions  $f$  with  $f \geq 0$  a.e. on the respective interval. Similarly, we say that  $f$  is positive if  $f \geq 0$  a.e. on the given interval.

**Proposition 2.3.1.** Fix  $0 < \beta \leq 1$ . Then we have the following assertions:

(i) The operators  $\mathbf{T}_\beta : L^1(I_1) \rightarrow L^1(I_1)$  and  $\mathcal{T}_\beta : L^1(I_1) \rightarrow L^1(I_1)$  are both norm contractions, which preserve the respective cones of positive functions.

(ii) On the positive functions,  $\mathcal{T}_\beta$  acts isometrically with respect to the  $L^1(I_1)$  norm.

(iii) If  $\mathcal{E}_{\beta, N}$  denotes the  $N$ -step wandering subset given by (2.3.1) above, then  $\mathbf{T}_\beta^N f = \mathcal{T}_\beta^N(1_{\mathcal{E}_{\beta, N}} f)$  for  $f \in L^1(I_1)$  and  $N = 1, 2, 3, \dots$

(iv) For  $0 < \beta < 1$ , and  $f \in L^1(I_1)$ , we have that  $\|\mathbf{T}_\beta^N f\|_{L^1(I_1)} \rightarrow 0$  as  $N \rightarrow +\infty$ . In particular,  $|\mathcal{E}_{\beta, N}| \rightarrow 0$  as  $N \rightarrow +\infty$ .

(v) For  $\beta = 1$  and  $f \in L^1(I_1)$  with mean  $\langle f, 1 \rangle_{I_1} = 0$ , we have that  $\|\mathbf{T}_1^N f\|_{L^1(I_1)} \rightarrow 0$  as  $N \rightarrow +\infty$ .

(vi) For  $\beta = 1$  and  $f \in L^1(I_1)$ , we have that  $\|1_{I_\eta} \mathbf{T}_1^N f\|_{L^1(I_1)} \rightarrow 0$  as  $N \rightarrow +\infty$  for each real  $\eta$  with  $0 < \eta < 1$ .

This is a conglomerate of ingredients from Propositions 3.4.1, 3.10.1, 3.11.3, 3.13.1, 3.13.2, and 3.13.3 in [12].

**2.4. An elementary observation extending the domain of definition for  $\mathbf{T}_\beta$ .** We begin with the following elementary observation.

**OBSERVATION.** The subtransfer and transfer operators  $\mathbf{T}_\beta$  and  $\mathcal{T}_\beta$ , initially defined on  $L^1$  functions, make sense for wider classes of functions. Indeed, if  $f \geq 0$ , then the formulae (1.2.3) and (1.2.5) make sense pointwise, with values in the extended nonnegative reals  $[0, +\infty]$ . More generally, if  $f$  is complex-valued, we may use the triangle inequality to dominate the convergence of  $\mathbf{T}_\beta f$  by that of  $\mathbf{T}_\beta |f|$ . This entails that  $\mathbf{T}_\beta f$  is well-defined a.e. if  $\mathbf{T}_\beta |f| < +\infty$  holds a.e. The same goes for  $\mathcal{T}_\beta$  of course.

This means that  $\mathbf{T}_\beta f$  will be well-defined for many functions  $f$ , not necessarily in  $L^1(I_1)$ .

**2.5. Symmetry preservation of the subtransfer operator  $\mathbf{T}_\beta$ .** The property that  $\mathbf{T}_\beta$  preserves symmetry on  $L^1(I_1)$  holds much more generally.

**Proposition 2.5.1.** *Fix  $0 < \beta \leq 1$ . To the extent that  $\mathbf{T}_\beta f$  is well-defined pointwise, we have the following:*

- (i) *If  $f$  is odd, then  $\mathbf{T}_\beta f$  is odd as well.*
- (ii) *If  $f$  is even, then  $\mathbf{T}_\beta f$  is even as well.*

This follows from Proposition 3.6.1 in [12].

Along with the symmetry, we can add constraints like monotonicity and convexity. Under such restraints on  $f$ , the pointwise values of  $\mathbf{T}_\beta f$  are guaranteed to exist, and the constraint is preserved under  $\mathbf{T}_\beta$ .

**Proposition 2.5.2.** *Fix  $0 < \beta \leq 1$ . We have the following:*

- (i) *If  $f : I_1 \rightarrow \mathbb{R}$  is odd and (strictly) increasing, then so is  $\mathbf{T}_\beta f$ .*
- (ii) *If  $f : I_1 \rightarrow \mathbb{R}$  is even and convex, and if  $f \geq 0$ , then so is  $\mathbf{T}_\beta f$ .*

This follows from Propositions 3.7.1 and 3.7.2 in [12].

**2.6. Preservation of point values of continuous functions under  $\mathbf{T}_\beta$ .** For  $\gamma$  with  $0 < \gamma < +\infty$ , let  $C(\bar{I}_\gamma)$  denote the space of continuous functions on the compact symmetric interval  $\bar{I}_\gamma = [-\gamma, \gamma]$ .

**Proposition 2.6.1.** *Fix  $0 < \beta \leq 1$ . If  $f \in C(\bar{I}_\beta)$ , then  $\mathbf{T}_\beta f \in C(\bar{I}_1)$ . Moreover, if in addition,  $f$  is odd, then  $\mathbf{T}_\beta f(1) = \beta f(\beta)$ .*

This result combines Propositions 3.8.1 and 3.8.2 in [12].

**2.7. Subinvariance of certain key functions.** Next, we consider the  $\mathbf{T}_\beta$ -iterates of the function

$$(2.7.1) \quad \kappa_\alpha(x) := \frac{\alpha}{\alpha^2 - x^2}, \quad x \in I_1,$$

where  $\alpha$  is assumed confined to the interval  $0 < \alpha \leq 1$ . This function is not in  $L^1(I_1)$ , although it is in  $L^{1,\infty}(I_1)$ . However, by the observation made in Subsection 2.4, we may still calculate the expression  $\mathbf{T}_\beta \kappa_\alpha$  pointwise wherever  $\mathbf{T}_\beta |\kappa_\alpha|(x) < +\infty$ . Note that  $\kappa_1(x)dx$  is the invariant measure for the transformation  $\tau_1(x) = \{-1/x\}_2$ , which in terms of the transfer operator  $\mathbf{T}_1$  means that  $\mathbf{T}_1 \kappa_1 = \kappa_1$ .

**Lemma 2.7.1.** *Fix  $0 < \beta \leq 1$ . For the function  $\kappa_\beta(x) = \beta/(\beta^2 - x^2)$ , we have that*

$$\mathbf{T}_\beta \kappa_\beta(x) = \mathbf{T}_\beta |\kappa_\beta|(x) = \kappa_1(x) = \frac{1}{1 - x^2}, \quad \text{a.e. } x \in I_1,$$

As for the function  $\kappa_1(x) = (1 - x^2)^{-1}$ , we have the estimate

$$0 \leq \mathbf{T}_\beta^n \kappa_1(x) \leq \beta^n \kappa_1(x) = \frac{\beta^n}{1 - x^2}, \quad x \in I_1, \quad n = 1, 2, 3, \dots,$$



which for  $0 < \beta < 1$ , may be replaced by the uniform estimate

$$\mathbf{T}_\beta^n \kappa_1(x) \leq \frac{2\beta^n}{1-\beta}, \quad x \in I_1, \quad n = 1, 2, 3, \dots$$

*Remark 2.7.2.* As noted earlier, for  $\beta = 1$ , we have the equality  $\mathbf{T}_1 \kappa_1 = \kappa_1$ .

### 3. BACKGROUND MATERIAL: THE HILBERT TRANSFORM ON THE LINE AND RELATED SPACES

**3.1. The Szegő projections and the Hardy  $H^1$ -space.** For a reference on the basic facts of Hardy spaces and BMO (bounded mean oscillation), we refer to, e.g., the monographs of Duren and Garnett [7], [10], as well as those of Stein [19], [20], and Stein and Weiss [21].

Let  $H_+^1(\mathbb{R})$  and  $H_-^1(\mathbb{R})$  be the subspaces of  $L^1(\mathbb{R})$  consisting of those functions whose Poisson extensions to the upper half plane

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

are holomorphic and conjugate-holomorphic, respectively. Here, we use the term conjugate-holomorphic (or anti-holomorphic) to mean that the complex conjugate of the function in question is holomorphic.

It is well-known that any function  $f \in H_+^1(\mathbb{R})$  has vanishing integral,

$$(3.1.1) \quad \langle f, 1 \rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t) dt = 0, \quad f \in H_+^1(\mathbb{R}).$$

In other words,  $H_+^1(\mathbb{C}) \subset L_0^1(\mathbb{R})$ , where

$$(3.1.2) \quad L_0^1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \langle f, 1 \rangle_{\mathbb{R}} = 0\}.$$

By a version of Liouville's theorem,

$$H_+^1(\mathbb{R}) \cap H_-^1(\mathbb{R}) = \{0\},$$

which allows us to think of the space

$$H_{\otimes}^1(\mathbb{R}) := H_+^1(\mathbb{R}) \oplus H_-^1(\mathbb{R})$$

as a linear subspace of  $L_0^1(\mathbb{R})$ . We will call  $H_{\otimes}^1(\mathbb{R})$  the *real  $H^1$ -space of the line  $\mathbb{R}$* , although it is  $\mathbb{C}$ -linear and the elements are generally complex-valued. It is not difficult to show that  $H_{\otimes}^1(\mathbb{R})$  is norm dense as a subspace of  $L_0^1(\mathbb{R})$ . The elements of  $f \in H_{\otimes}^1(\mathbb{R})$  are just the functions  $f \in L_0^1(\mathbb{R})$  which may be written in the form

$$(3.1.3) \quad f = f_1 + f_2, \quad \text{where } f_1 \in H_+^1(\mathbb{R}), \quad f_2 \in H_-^1(\mathbb{R}).$$

As already mentioned, the decomposition (3.1.3) is unique. As for notation, we let  $\mathbf{P}_+$  and  $\mathbf{P}_-$  denote the projections  $\mathbf{P}_+ f := f_1$  and  $\mathbf{P}_- f := f_2$  in the decomposition (3.1.3). These Szegő projections  $\mathbf{P}_+, \mathbf{P}_-$  can of course be extended beyond this  $H_{\otimes}^1(\mathbb{R})$  setting; more about this in the following subsection.

**3.2. The Hilbert and the modified Hilbert transform.** With respect to the dual action

$$\langle f, g \rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t) g(t) dt,$$

we may identify the dual space of  $H_{\otimes}^1(\mathbb{R})$  with  $\text{BMO}(\mathbb{R})/\mathbb{C}$ . Here,  $\text{BMO}(\mathbb{R})$  is the space of functions of *bounded mean oscillation*; this is the celebrated *Fefferman duality theorem* [8], [9]. As for notation, we write " $\cdot/\mathbb{C}$ " to express that we mod out with respect to the constant functions. One of the main results in the theory is the theorem of Fefferman and Stein [9] which tells us that

$$(3.2.1) \quad \text{BMO}(\mathbb{R}) = L^\infty(\mathbb{R}) + \tilde{\text{H}}L^\infty(\mathbb{R}).$$

or, in words, a function  $g$  is in  $\text{BMO}(\mathbb{R})$  if and only if it may be written in the form  $g = g_1 + \tilde{\mathbf{H}}g_2$ , where  $g_1, g_2 \in L^\infty(\mathbb{R})$ . Here,  $\tilde{\mathbf{H}}$  denotes the *modified Hilbert transform*, defined for  $f \in L^\infty(\mathbb{R})$  by the formula

$$(3.2.2) \quad \tilde{\mathbf{H}}f(x) := \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} f(t) \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} dt = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus [x-\epsilon, x+\epsilon]} f(t) \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} dt.$$

The decomposition (3.2.1) is clearly not unique. The non-uniqueness of the decomposition is equal to the intersection space

$$(3.2.3) \quad H_{\otimes}^\infty(\mathbb{R}) := L^\infty(\mathbb{R}) \cap \tilde{\mathbf{H}}L^\infty(\mathbb{R}),$$

which we refer to as the *real  $H^\infty$ -space*.

We should compare the modified Hilbert transform  $\tilde{\mathbf{H}}$  with the standard *Hilbert transform*  $\mathbf{H}$ , which acts boundedly on  $L^p(\mathbb{R})$  for  $1 < p < +\infty$ , and maps  $L^1(\mathbb{R})$  into  $L^{1,\infty}(\mathbb{R})$  for  $p = 1$ . Here,  $L^{1,\infty}(\mathbb{R})$  denotes the *weak- $L^1$  space*, see e.g. (1.2.9). The Hilbert transform of a function  $f$ , assumed integrable on the line  $\mathbb{R}$  with respect to the measure  $(1+t^2)^{-1/2}dt$ , is defined as the principal value integral

$$(3.2.4) \quad \mathbf{H}f(x) := \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} f(t) \frac{dt}{x-t} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{R} \setminus [x-\epsilon, x+\epsilon]} f(t) \frac{dt}{x-t}.$$

If  $f \in L^p(\mathbb{R})$ , where  $1 \leq p < +\infty$ , then both  $\mathbf{H}f$  and  $\tilde{\mathbf{H}}f$  are well-defined a.e., and it is easy to see that the difference  $\tilde{\mathbf{H}}f - \mathbf{H}f$  equals to a constant. It is often useful to think of the natural harmonic extensions of the Hilbert transforms  $\mathbf{H}f$  and  $\tilde{\mathbf{H}}f$  to the upper half-plane  $\mathbb{C}_+$  given by

$$(3.2.5) \quad \mathbf{H}f(z) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\text{Re } z - t}{|z - t|^2} f(t) dt, \quad \tilde{\mathbf{H}}f(z) := \frac{1}{\pi} \int_{\mathbb{R}} \left\{ \frac{\text{Re } z - t}{|z - t|^2} + \frac{t}{t^2 + 1} \right\} f(t) dt.$$

So, as a matter of normalization, we have that  $\tilde{\mathbf{H}}f(i) = 0$ . This tells us the value of the constant mentioned above:  $\tilde{\mathbf{H}}f - \mathbf{H}f = -\mathbf{H}f(i)$ .

Returning to the real  $H^1$ -space, we note the following characterization of the space in terms of the Hilbert transform: for  $f \in L^1(\mathbb{R})$ ,

$$(3.2.6) \quad f \in H_{\otimes}^1(\mathbb{R}) \iff f \in L_0^1(\mathbb{R}) \text{ and } \mathbf{H}f \in L_0^1(\mathbb{R}).$$

The Szegő projections  $\mathbf{P}_+$  and  $\mathbf{P}_-$  which were mentioned in Subsection 3.1 are more generally defined in terms of the Hilbert transform:

$$(3.2.7) \quad \mathbf{P}_+f := \frac{1}{2}(f + i\mathbf{H}f), \quad \mathbf{P}_-f := \frac{1}{2}(f - i\mathbf{H}f).$$

In a similar manner, for  $f \in L^\infty(\mathbb{R})$ , based on the modified Hilbert transform  $\tilde{\mathbf{H}}$  we may define the corresponding projections (which are actually projections modulo the constant functions)

$$(3.2.8) \quad \tilde{\mathbf{P}}_+f := \frac{1}{2}(f + i\tilde{\mathbf{H}}f), \quad \tilde{\mathbf{P}}_-f := \frac{1}{2}(f - i\tilde{\mathbf{H}}f),$$

so that, by definition,  $f = \tilde{\mathbf{P}}_+f + \tilde{\mathbf{P}}_-f$ .

#### 4. OPERATORS ON A SPACE OF DISTRIBUTIONS ON THE LINE

**4.1. The Hilbert transform on  $L^1$ .** For background material on the Hilbert transform and related topics, see, e.g. the monographs [7], [10], [19], [20], and [21].

It is well-known that the Hilbert transform as given by (3.2.4) maps  $\mathbf{H} : L^1(\mathbb{R}) \rightarrow L^{1,\infty}(\mathbb{R})$ . Since functions in  $L^{1,\infty}(\mathbb{R})$  have no obvious interpretation as distributions, it is better to define  $\mathbf{H}f$  right away as a distribution for  $f \in L^1(\mathbb{R})$ . The distributional interpretation is as follows:

$$(4.1.1) \quad \langle \varphi, \mathbf{H}f \rangle_{\mathbb{R}} := -\langle \mathbf{H}\varphi, f \rangle_{\mathbb{R}},$$

where  $\varphi$  is a test function with compact support, and  $f \in L^1(\mathbb{R})$ . Note that  $\mathbf{H}\varphi$ , the Hilbert transform of the test function, may be defined without the need of the principal value integral:

$$\mathbf{H}\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\varphi(x-t) - \varphi(x+t)}{t} dt;$$

it is a  $C^\infty$  function on  $\mathbb{R}$  with decay  $\mathbf{H}\varphi(x) = O(|x|^{-1})$  as  $|x| \rightarrow +\infty$ . As a consequence, it is clear from (4.1.1) how to extend the notion  $\mathbf{H}f$  to functions  $f$  with  $x \mapsto (1+x^2)^{-1/2}f(x)$  in  $L^1(\mathbb{R})$ . Note that as a result of the work of Kolmogorov, the equivalence (3.2.6) holds equally well when  $\mathbf{H}f$  is interpreted as a distribution and as a weak- $L^1$  function.

**4.2. The real  $H^\infty$  space.** The real  $H^\infty$  space is denoted by  $H^\infty_\otimes(\mathbb{R})$ , and it consists of all functions  $f \in L^\infty(\mathbb{R})$  of the form

$$(4.2.1) \quad f = f_1 + f_2, \quad f_1 \in H_+^\infty(\mathbb{R}), \quad f_2 \in H_-^\infty(\mathbb{R}).$$

Here,  $H_+^\infty(\mathbb{R})$  consists of all functions in  $L^\infty(\mathbb{R})$  whose Poisson extension to the upper half-plane is holomorphic, while  $H_-^\infty(\mathbb{R})$  consists of all functions in  $L^\infty(\mathbb{R})$  whose Poisson extension to the upper half-plane is conjugate-holomorphic (alternatively, the Poisson extension to the lower half-plane is holomorphic). The decomposition (4.2.1) is unique up to additive constants. It is easy to obtain the following equivalence, analogous to (3.2.6):

$$(4.2.2) \quad f \in H^\infty_\otimes(\mathbb{R}) \iff f, \tilde{\mathbf{H}}f \in L^\infty(\mathbb{R}).$$

**4.3. The predual of the real  $H^\infty$  space.** We shall be concerned with the following space of distributions on the line  $\mathbb{R}$ :

$$\mathfrak{V}(\mathbb{R}) := L^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R}),$$

which we supply with the appropriate norm (1.2.8), that is,

$$\|u\|_{\mathfrak{V}(\mathbb{R})} := \inf \left\{ \|f\|_{L^1(\mathbb{R})} + \|g\|_{L^1(\mathbb{R})} : u = f + \mathbf{H}g, \quad f \in L^1(\mathbb{R}), \quad g \in L_0^1(\mathbb{R}) \right\},$$

which makes  $\mathfrak{V}(\mathbb{R})$  a Banach space.

We recall that  $L_0^1(\mathbb{R})$  is the codimension-one subspace of  $L^1(\mathbb{R})$  which consists of the functions whose integral over  $\mathbb{R}$  vanishes. Given  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , the action of  $u := f + \mathbf{H}g$  on a test function  $\varphi$  is (compare with (4.1.1))

$$(4.3.1) \quad \langle \varphi, f + \mathbf{H}g \rangle_{\mathbb{R}} = \langle \varphi, f \rangle_{\mathbb{R}} - \langle \mathbf{H}\varphi, g \rangle_{\mathbb{R}} = \langle \varphi, f \rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}}\varphi, g \rangle_{\mathbb{R}};$$

we observe that the last identity uses that  $\langle 1, g \rangle_{\mathbb{R}} = 0$  and the fact that the functions  $\tilde{\mathbf{H}}\varphi$  and  $\mathbf{H}\varphi$  differ by a constant.

It remains to identify the dual space of  $\mathfrak{V}(\mathbb{R})$  with  $H^\infty_\otimes(\mathbb{R})$ .

**Proposition 4.3.1.** *Each continuous linear functional  $\mathfrak{V}(\mathbb{R}) \rightarrow \mathbb{C}$  corresponds to a function  $\varphi \in H^\infty_\otimes(\mathbb{R})$  in accordance with (4.3.1). In short, the dual space of  $\mathfrak{V}(\mathbb{R})$  equals  $H^\infty_\otimes(\mathbb{R})$ .*

This is Proposition 7.3.1 in [12]. We will refer to  $\mathfrak{V}(\mathbb{R})$  as the (canonical) *predual of the real  $H^\infty$  space*.

*Remark 4.3.2.* Since an  $L^1$ -function  $f$  gives rise to an absolutely continuous measure  $f(t)dt$ , it is natural to think of  $\mathfrak{V}(\mathbb{R})$  as embedded into the space  $\mathfrak{M}(\mathbb{R}) := M(\mathbb{R}) + \mathbf{H}M_0(\mathbb{R})$ , where  $M(\mathbb{R})$  denotes the space of complex-valued finite Borel measures on  $\mathbb{R}$ , and  $M_0(\mathbb{R})$  is the subspace of measures  $\mu \in M(\mathbb{R})$  with  $\mu(\mathbb{R}) = 0$ . The Hilbert transforms of singular measures noticeably differ from those of absolutely continuous measures (see [18]).

**4.4. The “valeur au point” function associated with an element of  $\mathfrak{V}(\mathbb{R})$ .** We recall that  $\mathfrak{V}(\mathbb{R})$  consists of distributions on the real line. However, the definition

$$\mathfrak{V}(\mathbb{R}) = L^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R})$$

would allow us to also think of this space as a subspace of  $L^{1,\infty}(\mathbb{R})$ , the weak  $L^1$ -space. It is a natural question to wonder about the relationship between the distribution and the  $L^{1,\infty}$  function. We stick with the distribution theory definition of  $\mathfrak{V}(\mathbb{R})$ , and associate with a given  $u \in \mathfrak{V}(\mathbb{R})$  the “valeur au point” function  $\text{vp}[u]$  at almost all points of the line. The precise definition of  $\text{vp}[u]$  is as follows.

**Definition 4.4.1.** For a fixed  $x \in \mathbb{R}$ , let  $\chi = \chi_x$  is a compactly supported  $C^\infty$ -smooth function on  $\mathbb{R}$  with  $\chi(t) = 1$  for all  $t$  in an open neighborhood of the point  $x$ . Also, let

$$P_{x+i\epsilon}(t) := \pi^{-1} \frac{\epsilon}{\epsilon^2 + (x-t)^2}$$

be the Poisson kernel. The *valeur au point* function associated with the distribution  $u$  on  $\mathbb{R}$  is the function  $\text{vp}[u] = \text{vp}[u\chi]$  given by

$$(4.4.1) \quad \text{vp}[u](x) := \lim_{\epsilon \rightarrow 0^+} \langle \chi P_{x+i\epsilon}, u \rangle_{\mathbb{R}}, \quad x \in \mathbb{R},$$

wherever the limit exists.

In principle,  $\text{vp}[u](x)$  might depend on the choice of the cut-off function  $\chi$ . Lemma 7.4.2 in [12] guarantees that this is not the case, and that almost everywhere, it gives the same result as the weak- $L^1$  interpretation of the Hilbert transform on  $L^1(\mathbb{R})$ . A basic result is the following.

**Proposition 4.4.2.** (Kolmogorov) *The mapping  $\text{vp} : \mathfrak{L}(\mathbb{R}) \rightarrow L^{1,\infty}(\mathbb{R})$ ,  $u \mapsto \text{vp}[u]$ , is injective and continuous.*

This is a combination of Propositions 7.4.3 and 7.4.4 in [12].

**4.5. The restriction of  $\mathfrak{L}(\mathbb{R})$  to an interval.** If  $u$  is a distribution on an open interval  $J$ , then the restriction of  $u$  to an open subinterval  $I$ , denoted  $u|_I$ , is the distribution defined by

$$\langle \varphi, u|_I \rangle := \langle \varphi, u \rangle_I,$$

where  $\varphi$  is a  $C^\infty$ -smooth test function whose support is compact and contained in  $I$ .

**Definition 4.5.1.** Let  $I$  be an open interval of the real line. Then  $u \in \mathfrak{L}(I)$  means by definition that  $u$  is a distribution on  $I$  such that there exists a distribution  $v \in \mathfrak{L}(\mathbb{R})$  such that  $u = v|_I$ .

Kolmogorov's theorem (Proposition 4.4.2) has a local version as well.

**Proposition 4.5.2.** (Kolmogorov) *Let  $I$  be a nonempty open interval of the line  $\mathbb{R}$ . Then the “valeur au point” mapping is injective and continuous  $\text{vp} : \mathfrak{L}(I) \rightarrow L^{1,\infty}(I)$ .*

This is a combination of Corollaries 7.6.3 and 7.6.6 in [12].

## 5. BACKGROUND MATERIAL: FUNCTION SPACES ON THE CIRCLE

**5.1. The Hardy space  $H^1$  on the circle.** Let  $L^1(\mathbb{R}/2\mathbb{Z})$  denote the space of (equivalence classes of) 2-periodic Borel measurable functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  subject to the integrability condition

$$\|f\|_{L^1(\mathbb{R}/2\mathbb{Z})} := \int_{I_1} |f(t)| dt < +\infty,$$

where  $I_1 = ]-1, 1[$  as before. Via the exponential mapping  $t \mapsto e^{i\pi t}$ , which is 2-periodic and maps the real line  $\mathbb{R}$  onto the unit circle  $\mathbb{T}$ , we may identify the space  $L^1(\mathbb{R}/2\mathbb{Z})$  with the standard Lebesgue space  $L^1(\mathbb{T})$  of the unit circle. This will allow us to develop the elements of Hardy space theory in the setting of 2-periodic functions. We shall need the subspace  $L_0^1(\mathbb{R}/2\mathbb{Z})$  consisting of all  $f \in L^1(\mathbb{R}/2\mathbb{Z})$  with

$$\langle f, 1 \rangle_{I_1} = \int_{I_1} f(t) dt = 0;$$

it has codimension 1 in  $L^1(\mathbb{R}/2\mathbb{Z})$ . The Hardy space  $H_+^1(\mathbb{R}/2\mathbb{Z})$  is defined as the subspace of  $L^1(\mathbb{R}/2\mathbb{Z})$  consisting of functions  $g \in L^1(\mathbb{R}/2\mathbb{Z})$  whose Poisson extension to the unit disk  $\mathbb{D}$  is holomorphic and vanishes at the origin, and analogously,  $H_-^1(\mathbb{R}/2\mathbb{Z})$  consists of the functions  $g$  in  $L^1(\mathbb{R}/2\mathbb{Z})$  whose complex conjugate  $\bar{g}$  is in  $H_+^1(\mathbb{R}/2\mathbb{Z})$ . In terms of the Poisson extensions to the upper half-plane instead,  $f \in H_+^1(\mathbb{R}/2\mathbb{Z})$  if the extension is holomorphic and vanishes at  $+i\infty$ , whereas  $f \in H_-^1(\mathbb{R}/2\mathbb{Z})$  if the extension is conjugate-holomorphic and vanishes at  $+i\infty$ . We then introduce the *real  $H^1$ -space*

$$H_{\otimes}^1(\mathbb{R}/2\mathbb{Z}) := H_+^1(\mathbb{R}/2\mathbb{Z}) \oplus H_-^1(\mathbb{R}/2\mathbb{Z}),$$

where we think of the elements of the sum space as 2-periodic functions on  $\mathbb{R}$  (as before the symbol  $\oplus$  means direct sum, which is possible since  $H_+^1(\mathbb{R}/2\mathbb{Z}) \cap H_-^1(\mathbb{R}/2\mathbb{Z}) = \{0\}$ ). We note that, for instance,  $H_\oplus^1(\mathbb{R}/2\mathbb{Z}) \subset L_0^1(\mathbb{R}/2\mathbb{Z})$ .

**5.2. The Hilbert transform on 2-periodic functions and distributions.** For  $f \in L^1(\mathbb{R}/2\mathbb{Z})$ , we let  $\mathbf{H}_2$  be the convolution operator

$$(5.2.1) \quad \mathbf{H}_2 f(x) := \frac{1}{2} \text{pv} \int_{I_1} f(t) \cot \frac{\pi(x-t)}{2} dt,$$

where again pv stands for principal value, which means we take the limit as  $\epsilon \rightarrow 0^+$  of the integral where the set

$$\{x\} + 2\mathbb{Z} + [-\epsilon, \epsilon]$$

is removed from the interval  $I_1 = ]-1, 1[$ . It is obvious from the periodicity of the cotangent function that  $\mathbf{H}_2 f$ , if it exists as a limit, is 2-periodic. Alternatively, by a change of variables, we have that

$$(5.2.2) \quad \mathbf{H}_2 f(x) = \frac{1}{2} \lim_{\epsilon \rightarrow 0^+} \int_{I_1 \setminus I_\epsilon} f(x-t) \cot \frac{\pi t}{2} dt,$$

(here, as usual,  $I_\epsilon = ]-\epsilon, \epsilon[$ ). It is well-known that the operator  $\mathbf{H}_2$  is just the natural extension of the Hilbert transform  $\mathbf{H}$  to the 2-periodic functions. We observe the peculiarity that  $\mathbf{H}_2 1 = 0$ , which follows from the fact that the cotangent function is odd. Like the situation for the real line  $\mathbb{R}$ , the periodic Hilbert transform  $\mathbf{H}_2$  maps  $L^1(\mathbb{R}/2\mathbb{Z})$  into the weak  $L^1$ -space  $L^{1,\infty}(\mathbb{R}/2\mathbb{Z})$ . However, as we prefer to work within the framework of distribution theory, we proceed as follows.

Let  $C^\infty(\mathbb{R}/2\mathbb{Z})$  denote the space of  $C^\infty$ -smooth 2-periodic functions on  $\mathbb{R}$ . It is easy to see that

$$\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z}) \implies \mathbf{H}_2 \varphi \in C^\infty(\mathbb{R}/2\mathbb{Z}).$$

To emphasize the importance of the circle  $\mathbb{T} \cong \mathbb{R}/2\mathbb{Z}$ , we write

$$(5.2.3) \quad \langle f, g \rangle_{\mathbb{R}/2\mathbb{Z}} := \int_{-1}^1 f(t)g(t)dt,$$

for the dual action when  $f$  and  $g$  are 2-periodic.

**Definition 5.2.1.** For a test function  $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$  and a distribution  $u$  on the circle  $\mathbb{R}/2\mathbb{Z}$ , we put

$$\langle \varphi, \mathbf{H}_2 u \rangle_{\mathbb{R}/2\mathbb{Z}} := -\langle \mathbf{H}_2 \varphi, u \rangle_{\mathbb{R}/2\mathbb{Z}}.$$

This defines the Hilbert transform  $\mathbf{H}_2 u$  for any distribution  $u$  on the circle  $\mathbb{R}/2\mathbb{Z}$ .

**5.3. The real  $H^\infty$ -space of the circle.** The real  $H^\infty$ -space on the circle  $\mathbb{R}/2\mathbb{Z}$  is denoted by  $H_\oplus^\infty(\mathbb{R}/2\mathbb{Z})$ , and consists of all the functions in  $H_\oplus^\infty(\mathbb{R})$  that are 2-periodic. It follows from (4.2.2) that

$$(5.3.1) \quad f \in H_\oplus^\infty(\mathbb{R}/2\mathbb{Z}) \iff f, \mathbf{H}_2 f \in L^\infty(\mathbb{R}/2\mathbb{Z}).$$

**5.4. A predual of 2-periodic real  $H^\infty$ .** We put

$$\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) := L^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L_0^1(\mathbb{R}/2\mathbb{Z}),$$

understood as a space of 2-periodic distributions on the line  $\mathbb{R}$ . More precisely, if  $u = f + \mathbf{H}_2 g$ , where  $f \in L^1(\mathbb{R}/2\mathbb{Z})$  and  $g \in L_0^1(\mathbb{R}/2\mathbb{Z})$ , then the action on a test function  $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$  is given by

$$(5.4.1) \quad \langle \varphi, u \rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \varphi, f \rangle_{\mathbb{R}/2\mathbb{Z}} - \langle \mathbf{H}_2 \varphi, g \rangle_{\mathbb{R}/2\mathbb{Z}}.$$

But a 2-periodic distribution should be possible to think of as a distribution on the line, which means that need to understand the action on standard test functions in  $C_c^\infty(\mathbb{R})$ . If  $\psi \in C_c^\infty(\mathbb{R})$ , we simply put

$$(5.4.2) \quad \langle \psi, u \rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \Pi_2 \psi, u \rangle_{\mathbb{R}/2\mathbb{Z}},$$

where  $\Pi_2\psi \in C^\infty(\mathbb{R}/2\mathbb{Z})$  is given by

$$(5.4.3) \quad \Pi_2\psi(x) := \sum_{j \in \mathbb{Z}} \psi(x + 2j).$$

We will refer to  $\Pi_2$  as the *periodization operator*.

As in the case of the line  $\mathbb{R}$ , we may identify  $\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  with the predual of the real  $H^\infty$ -space  $H^\infty_\otimes(\mathbb{R}/2\mathbb{Z})$ :

$$\mathfrak{L}(\mathbb{R}/2\mathbb{Z})^* = H^\infty_\otimes(\mathbb{R}/2\mathbb{Z})$$

with respect to the standard dual action  $\langle \cdot, \cdot \rangle_{\mathbb{R}/2\mathbb{Z}}$ .

The definition of the “valeur au point function”  $\text{vp}[u]$  makes sense for  $u \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  and as in the case of the line, it does not depend on the choice of the particular cut-off function. The following assertion is the analogue of Proposition 4.4.2; the proof is suppressed.

**Proposition 5.4.1.** (Kolmogorov) *The “valeur au point” mapping  $\text{vp} : \mathfrak{L}(\mathbb{R}/2\mathbb{Z}) \rightarrow L^{1,\infty}(\mathbb{R}/2\mathbb{Z})$ ,  $u \mapsto \text{vp}[u]$ , is injective and continuous.*

## 6. A SUM OF TWO PREDUALS AND ITS LOCALIZATION TO INTERVALS

**6.1. The sum space  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ .** Suppose  $u$  is distribution on the line  $\mathbb{R}$  of the form

$$(6.1.1) \quad u = v + w, \quad \text{where } v \in \mathfrak{L}(\mathbb{R}), \quad w \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z}).$$

The natural question appears as to whether the distributions  $v, w$  on the right-hand side are unique for a given  $u$ . This is indeed so (Proposition 9.1.1 in [12]):

$$(6.1.2) \quad \mathfrak{L}(\mathbb{R}) \cap \mathfrak{L}(\mathbb{R}/2\mathbb{Z}) = \{0\}.$$

In view of (6.1.2), it makes sense to write  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  for the space of tempered distributions  $u$  of the form (6.1.1). We endow  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  with the induced Banach space norm

$$\|u\|_{\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})} := \|v\|_{\mathfrak{L}(\mathbb{R})} + \|w\|_{\mathfrak{L}(\mathbb{R}/2\mathbb{Z})},$$

provided  $u, v, w$  are related via (6.1.1).

**6.2. The localization of  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  to a bounded open interval.** In the sense of Subsection 4.5, we may restrict a given distribution  $u \in \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  to a given open interval  $I$ . It is natural to wonder what the space of such restrictions looks like.

**Proposition 6.2.1.** *The restriction of the space  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  to a bounded open interval  $I$  equals the space  $\mathfrak{L}(I)$ .*

This is Proposition 9.2.1 in [12].

## 7. AN INVOLUTION, ITS ADJOINT, AND THE PERIODIZATION OPERATOR

**7.1. An involutive operator.** For each positive real number  $\beta$ , let  $\mathbf{J}_\beta$  denote the involution given by

$$\mathbf{J}_\beta f(x) := \frac{\beta}{x^2} f(-\beta/x), \quad x \in \mathbb{R}^\times.$$

We use the standard notation  $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ . If  $f \in L^1(\mathbb{R})$  and  $\varphi \in L^\infty(\mathbb{R})$ , the change-of-variables formula yields that

$$(7.1.1) \quad \langle \varphi, \mathbf{J}_\beta f \rangle_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(t) f(-\beta/t) \frac{\beta dt}{t^2} = \int_{\mathbb{R}} \varphi(-\beta/t) f(t) dt = \langle \mathbf{J}_\beta^* \varphi, f \rangle_{\mathbb{R}},$$

where  $\mathbf{J}_\beta^*$  is the involution

$$\mathbf{J}_\beta^* \varphi(t) := \varphi(-\beta/t), \quad t \in \mathbb{R}^\times.$$

It is a consequence of the change-of-variables formula that  $\mathbf{J}_\beta$  is an isometric isomorphism  $L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ .

Next, we extend  $\mathbf{J}_\beta$  to a bounded operator  $\mathfrak{L}(\mathbb{R}) \rightarrow \mathfrak{L}(\mathbb{R})$ . The arguments in Subsection 10.1 of [12] show that the correct extension of  $\mathbf{J}_\beta$  to an operator  $\mathfrak{L}(\mathbb{R}) \rightarrow \mathfrak{L}(\mathbb{R})$  reads as follows.

**Definition 7.1.1.** For  $u \in \mathfrak{L}(\mathbb{R})$  of the form  $u = f + \mathbf{H}g \in \mathfrak{L}(\mathbb{R})$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , we define the  $\mathbf{J}_\beta u$  to be the distribution on  $\mathbb{R}$  given by the formula

$$\langle \varphi, \mathbf{J}_\beta u \rangle_{\mathbb{R}} = \langle \varphi, \mathbf{J}_\beta(f + \mathbf{H}g) \rangle_{\mathbb{R}} := \langle \varphi, \mathbf{J}_\beta f \rangle_{\mathbb{R}} + \langle \varphi, \mathbf{H} \mathbf{J}_\beta g \rangle_{\mathbb{R}} = \langle \varphi, \mathbf{J}_\beta f \rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}} \varphi, \mathbf{J}_\beta g \rangle_{\mathbb{R}},$$

for test functions  $\varphi \in H_{\otimes}^\infty(\mathbb{R})$ .

The involutive properties of  $\mathbf{J}_\beta$  and its adjoint are then naturally preserved (Proposition 10.1.4 in [12]).

**7.2. The periodization operator.** We recall the definition of the *periodization operator*  $\Pi_2$ :

$$\Pi_2 f(x) := \sum_{j \in \mathbb{Z}} f(x + 2j).$$

In (5.4.3), we defined the  $\Pi_2$  on test functions. It is however clear that it remains well-defined with much less smoothness required of  $f$ . The terminology comes from the property that whenever it is well-defined, the function  $\Pi_2 f$  is 2-periodic automatically. It is obvious from the definition that  $\Pi_2$  acts contractively  $L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R}/2\mathbb{Z})$ .

The basic property of the periodization operator is the following, for  $f \in L^1(\mathbb{R})$  and  $F \in L^\infty(\mathbb{R}/2\mathbb{Z})$  (see, e.g., (10.2.2) in [12]):

$$(7.2.1) \quad \langle F, \Pi_2 f \rangle_{\mathbb{R}/2\mathbb{Z}} = \langle F, f \rangle_{\mathbb{R}}, \quad n \in \mathbb{Z}.$$

We need to extend  $\Pi_2$  in a natural fashion to the space  $\mathfrak{L}(\mathbb{R})$ . If  $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$  is a test function on the circle, we glance at (7.2.1), and for  $u \in \mathfrak{L}(\mathbb{R})$  with  $u = f + \mathbf{H}g$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , we set

$$(7.2.2) \quad \langle \varphi, \Pi_2 u \rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \varphi, u \rangle_{\mathbb{R}} = \langle \varphi, f \rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}} \varphi, g \rangle_{\mathbb{R}}.$$

This defines  $\Pi_2 u$  as a distribution on the circle (compare with (4.3.1)).

**Proposition 7.2.1.** For  $u \in \mathfrak{L}(\mathbb{R})$  of the form  $u = f + \mathbf{H}g$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , we have that  $\Pi_2 u = \Pi_2 f + \mathbf{H}_2 \Pi_2 g$ . In particular,  $\Pi_2$  maps  $\mathfrak{L}(\mathbb{R}) \rightarrow \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  continuously.

This is Proposition 10.2.2 in [12].

## 8. REFORMULATION OF THE SPANNING PROBLEM OF THEOREM 1.3.1

**8.1. An equivalence.** Let us write  $\mathbb{Z}_{+,0} := \{0, 1, 2, \dots\}$ .

**Lemma 8.1.1.** Fix  $0 < \beta < +\infty$ . Then the following conditions (a) and (b) are equivalent:

(a) The linear span of the functions

$$e_n(t) := e^{i\pi n t}, \quad e_m^{(\beta)}(t) := e^{-i\pi \beta m/t}, \quad m, n \in \mathbb{Z}_{+,0},$$

is weak-star dense in  $H_+^\infty(\mathbb{R})$ .

(b) For  $f \in L_0^1(\mathbb{R})$ , the following implication holds:

$$\Pi_2 f, \Pi_2 \mathbf{J}_\beta f \in H_+^1(\mathbb{R}/2\mathbb{Z}) \implies f \in H_+^1(\mathbb{R}).$$

We remark that the functions  $e^{i\pi n t}$  and  $e^{-i\pi \beta m/t}$  for  $m, n \in \mathbb{Z}_{+,0}$  belong to  $H_+^\infty(\mathbb{R})$  (after all, they have bounded holomorphic extensions to  $\mathbb{C}_+$ ), so that part (a) makes sense.

*Proof of Lemma 8.1.1.* With respect to the dual action  $\langle \cdot, \cdot \rangle_{\mathbb{R}}$  on the line, the predual of  $H_+^\infty(\mathbb{R})$  is the quotient space  $L^1(\mathbb{R})/H_+^1(\mathbb{R})$ . With this in mind, the assertion of part (a) is seen to be equivalent to the following: For any  $f \in L^1(\mathbb{R})$ , the implication

$$(8.1.1) \quad \left\{ \forall m, n \in \mathbb{Z}_{+,0} : \langle e_n, f \rangle_{\mathbb{R}} = \langle e_m^{(\beta)}, f \rangle_{\mathbb{R}} = 0 \right\} \implies f \in H_+^1(\mathbb{R})$$

holds. By testing with e.g.  $n = 0$ , we note that we might as well assume that  $f \in L_0^1(\mathbb{R})$  in (8.1.1). By the basic property (7.2.1) of the periodization operator  $\Pi_2$ , we have that

$$(8.1.2) \quad \langle e_n, f \rangle_{\mathbb{R}} = \langle e_n, \Pi_2 f \rangle_{\mathbb{R}/2\mathbb{Z}},$$

from which we conclude that

$$\left\{ \forall n \in \mathbb{Z}_{+,0} : \langle e_n, f \rangle_{\mathbb{R}} = 0 \right\} \iff \Pi_2 f \in H_+^1(\mathbb{R}/2\mathbb{Z}).$$

Since  $\mathbf{J}_\beta^* e_m = e_m^{(\beta)}$ , where  $\mathbf{J}_\beta^*$  is the involution studied in Subsection 7.1, a repetition of the above gives that for  $f \in L_0^1(\mathbb{R})$ , we have the equivalence

$$\left\{ \forall m \in \mathbb{Z}_{+,0} : \langle e_m^{(\beta)}, f \rangle_{\mathbb{R}} = 0 \right\} \iff \Pi_2 \mathbf{J}_\beta f \in H_+^1(\mathbb{R}/2\mathbb{Z}).$$

By splitting the annihilation conditions in (8.1.1), we see that they are equivalent to having both  $\Pi_2 f$  and  $\Pi_2 \mathbf{J}_\beta f$  in  $H_+^1(\mathbb{R}/2\mathbb{Z})$ . In other words, conditions (a) and (b) are equivalent.  $\square$

*Remark 8.1.2.* By the argument involving point separation in  $\mathbb{C}_+$  from [11], the condition  $\beta \leq 1$  is necessary for part (a) of Lemma 8.1.1 to hold. Actually, as mentioned in the introduction, the methods of [5] supply infinitely many linearly independent counterexamples for  $\beta > 1$ .

*Remark 8.1.3.* If we think of  $\Pi_2 f$  and  $\Pi_2 \mathbf{J}_\beta f$  as 2-periodic “shadows” of  $f$  and  $\mathbf{J}_\beta f$ , the issue at hand in part (b) of Lemma 8.1.1 is whether knowing that the two shadows are in the right space we may conclude the function comes from the space  $H_+^1(\mathbb{R})$ . We note here that the main result of [11] may be understood as the assertion that  $f$  is determined uniquely by the two “shadows”  $\Pi_2 f$  and  $\Pi_2 \mathbf{J}_\beta f$  if and only if  $\beta \leq 1$ .

**8.2. An alternative statement in terms of the space  $\mathfrak{L}(\mathbb{R})$ .** Let  $\mathfrak{L}_0(\mathbb{R})$  denote the space

$$\mathfrak{L}_0(\mathbb{R}) := L_0^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}),$$

which has codimension 1 in  $\mathfrak{L}(\mathbb{R})$ .

**Lemma 8.2.1.** Fix  $0 < \beta \leq 1$ . Then (a)  $\implies$  (b), where (a) and (b) are the following assertions:

(a) For  $u \in \mathfrak{L}_0(\mathbb{R})$ , the following implication holds:

$$\Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0 \implies u = 0.$$

(b) For  $f \in L_0^1(\mathbb{R})$ , the following implication holds:

$$\Pi_2 f, \Pi_2 \mathbf{J}_\beta f \in H_+^1(\mathbb{R}/2\mathbb{Z}) \implies f \in H_+^1(\mathbb{R}).$$

*Proof.* We connect  $u \in \mathfrak{L}_0(\mathbb{R})$  with  $f \in L_0^1(\mathbb{R})$  via the conjugate-holomorphic Szegő projection  $u := \mathbf{P}_- f = \frac{1}{2}(f - i\mathbf{H}f)$ . If  $\Pi_2 f \in H_+^1(\mathbb{R}/2\mathbb{Z})$ , then by a Liouville-type argument,  $\Pi_2 u = 0$  holds. Analogously, if  $\Pi_2 \mathbf{J}_\beta f \in H_+^1(\mathbb{R}/2\mathbb{Z})$ , then we obtain that  $\Pi_2 \mathbf{J}_\beta u = 0$ . So, from the implication of part (a), we obtain from the assumptions in (b) that  $u = 0$ , that is, that  $f \in H_+^1(\mathbb{R})$ . This means that the implication of (a) implies that of (b), as claimed.  $\square$

*Remark 8.2.2.* Condition (b) of Lemma 8.2.1 has acquired the same general appearance as in the analysis of the  $L^\infty(\mathbb{R})$  problem, but at the cost of considering the larger space  $\mathfrak{L}_0(\mathbb{R})$  in place of  $L_0^1(\mathbb{R})$ . This is unavoidable, as the weak-star topology of the real Hardy space  $H_\infty^\infty(\mathbb{R})$  is finer than that of  $L^\infty(\mathbb{R})$ . Our proof of Theorem 1.3.1 passes through Lemmas 8.1.1 and 8.2.1, and we ultimately show that the implication (a) of Lemma 8.2.1 is valid for  $0 < \beta \leq 1$ . It then follows from Lemmas 8.1.1 and 8.2.1 that assertion (a) of Lemma 8.1.1 is valid in the range  $0 < \beta \leq 1$ . In its turn, the proof that the implication (a) of Lemma 8.2.1 holds for  $0 < \beta \leq 1$  is based on an extension of ergodic theory for Gauss-type maps, developed in Sections 9–14.

## 9. A SUBTRANSFER OPERATOR ON A SPACE OF DISTRIBUTIONS

**9.1. Restrictions of  $\mathfrak{L}(\mathbb{R})$  to a symmetric interval and to its complement.** In Subsection 4.5, we defined the restriction of  $\mathfrak{L}(\mathbb{R})$  to an open interval. Here we the restriction to the complement of a closed interval as well. For a positive real parameter  $\gamma$ , we consider the symmetric interval  $I_\gamma$  and its closure  $\bar{I}_\gamma$  as in Subsection 2.1,

$$I_\gamma = ]-\gamma, \gamma[, \quad \bar{I}_\gamma = [-\gamma, \gamma].$$



We recall that by Definition 4.5.1, the space  $\mathfrak{L}(I_\gamma)$  is defined as

$$\mathfrak{L}(I_\gamma) := \{u \in \mathcal{D}'(I_\gamma) : \exists U \in \mathfrak{L}(\mathbb{R}) \text{ with } U|_{I_\gamma} = u\}$$

and analogously we may define  $\mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma)$  for the complementary interval  $\mathbb{R} \setminus \bar{I}_\gamma$ :

$$\mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma) := \{u \in \mathcal{D}'(\mathbb{R} \setminus \bar{I}_\gamma) : \exists U \in \mathfrak{L}(\mathbb{R}) \text{ with } U|_{\mathbb{R} \setminus \bar{I}_\gamma} = u\}.$$

Here,  $\mathcal{D}'$  has the standard interpretation of the space of Schwartzian distributions on the given interval. Of course, in the sense of distribution theory, taking the restriction to an open subset has the interpretation of considering the linear functional restricted to test functions supported on that given open subset. The norm on each of the spaces  $\mathfrak{L}(I_\gamma)$  and  $\mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma)$  is the associated quotient norm, where we mod out with respect to the distributions in  $\mathfrak{L}(\mathbb{R})$  whose support is contained in the complementary closed set (cf. Subsection 4.5).

We will need to work with restrictions to  $I_\gamma$  and  $\mathbb{R} \setminus \bar{I}_\gamma$  repeatedly, so it is good idea to introduce appropriate notation.

**Definition 9.1.1.** We let  $\mathbf{R}_\gamma$  denote the operation of restricting a distribution to the interval  $I_\gamma$ . Analogously, we let  $\mathbf{R}_\gamma^\dagger$  denote the operation of restricting a distribution to the open set  $\mathbb{R} \setminus \bar{I}_\gamma$ .

**9.2. The involution on the local spaces.** We need to understand the action of the involution  $\mathbf{J}_\beta$  defined in Subsection 7.1 on the local spaces  $\mathfrak{L}(I_\gamma)$  and  $\mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma)$ .

**Proposition 9.2.1.** Fix  $0 < \beta, \gamma < +\infty$ . The involution  $\mathbf{J}_\beta$  defines continuous maps

$$\mathbf{J}_\beta : \mathfrak{L}(I_\gamma) \rightarrow \mathfrak{L}(\mathbb{R} \setminus \bar{I}_{\beta/\gamma}) \quad \text{and} \quad \mathbf{J}_\beta : \mathfrak{L}(\mathbb{R} \setminus \bar{I}_\gamma) \rightarrow \mathfrak{L}(I_{\beta/\gamma}).$$

*Proof.* The assertion is rather immediate from the mapping properties of  $\mathbf{J}_\beta$  (see Subsection 7.1) and the localization procedure.  $\square$

**9.3. Splitting of the periodization operator.** We split the periodization operator  $\mathbf{\Pi}_2$  in two parts:  $\mathbf{\Pi}_2 = \mathbf{I} + \mathbf{\Sigma}_2$ , where  $\mathbf{I}$  is the identity and  $\mathbf{\Sigma}_2$  is the operator defined by

$$\mathbf{\Sigma}_2 u(x) := \sum_{j \in \mathbb{Z}^\times} u(x + 2j),$$

whenever the right-hand side is meaningful in the sense of distributions. Here, we use the notation  $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$ . In view of Proposition 7.2.1, the proof of the following proposition is immediate.

**Proposition 9.3.1.** The operator  $\mathbf{\Sigma}_2$  maps  $\mathfrak{L}(\mathbb{R}) \rightarrow \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  continuously.

**Definition 9.3.2.** Let  $\mathbf{\Sigma}_2^{(1)} : \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1) \rightarrow \mathfrak{L}(I_1)$  be defined as follows. Given a distribution  $u \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$ , we find a  $U \in \mathfrak{L}(\mathbb{R})$  whose restriction is  $\mathbf{R}_1 U = u$ . Then we put (use Proposition 6.2.1)

$$\mathbf{\Sigma}_2^{(1)} u := \mathbf{R}_1 \mathbf{\Sigma}_2 U \in \mathbf{R}_1(\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})) = \mathfrak{L}(I_1).$$

We will call  $\mathbf{\Sigma}_2^{(1)}$  the *compression* of  $\mathbf{\Sigma}_2$ . However, we still need to verify that this definition is consistent, that is, that the right-hand side  $\mathbf{R}_1 \mathbf{\Sigma}_2 U$  is independent of the choice of the extension  $U$ .

**Proposition 9.3.3.** The operator  $\mathbf{\Sigma}_2^{(1)} : \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1) \rightarrow \mathfrak{L}(I_1)$  is well-defined and bounded. Moreover, we have that  $\mathbf{R}_1 \mathbf{\Sigma}_2 U = \mathbf{\Sigma}_2^{(1)} \mathbf{R}_1^\dagger U$  holds for  $U \in \mathfrak{L}(\mathbb{R})$ .

*Proof.* To see that  $\mathbf{\Sigma}_2^{(1)}$  is well-defined, we need to check that if  $U \in \mathfrak{L}(\mathbb{R})$  and its restriction to  $\mathbb{R} \setminus \bar{I}_1$  vanishes (this means that  $\text{supp } U \subset \bar{I}_1$ ), then  $\mathbf{R}_1 \mathbf{\Sigma}_2 U = 0$ . From the definition of the operator  $\mathbf{\Sigma}_2$ , we understand that

$$\text{supp } \mathbf{\Sigma}_2 U \subset \text{supp } U + 2\mathbb{Z}^\times \subset \bar{I}_1 + 2\mathbb{Z}^\times = \mathbb{R} \setminus I_1.$$

In particular, the restriction to  $I_1$  of  $\Sigma_2 U$  vanishes, as required. Similarly, we argue that  $\Sigma_2^{\textcircled{1}}$  is bounded, based on Proposition 6.2.1 and Definition 9.3.2. Finally, the asserted identity  $\mathbf{R}_1 \Sigma_2 U = \Sigma_2^{\textcircled{1}} \mathbf{R}_1^{\dagger} U$  just expresses how the operator  $\Sigma_2^{\textcircled{1}}$  is defined.  $\square$

**9.4. Further analysis of the uniqueness problem.** The complementary restriction operators have the following properties:

$$(9.4.1) \quad \mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} u = \mathbf{J}_{\beta} \mathbf{R}_{\beta} u, \quad u \in \mathfrak{U}(\mathbb{R}),$$

and, for  $0 < \beta \leq \gamma < +\infty$ ,

$$(9.4.2) \quad \mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} u = \mathbf{J}_{\beta} \mathbf{R}_{\beta} u, \quad u \in \mathfrak{U}(I_{\gamma}).$$

They will help us analyze further the tentative implication (a) of Lemma 8.2.1.

**Proposition 9.4.1.** *Fix  $0 < \beta \leq 1$ . Suppose that for  $u \in \mathfrak{U}(\mathbb{R})$  we have  $\Pi_2 u = 0$  and  $\Pi_2 \mathbf{J}_{\beta} u = 0$ . Then the restrictions  $u_0 := \mathbf{R}_1 u \in \mathfrak{U}(I_1)$  and  $u_1 := \mathbf{R}_1^{\dagger} u \in \mathfrak{U}(\mathbb{R} \setminus \bar{I}_1)$  each solve the equations*

$$u_0 = \Sigma_2^{\textcircled{1}} \mathbf{J}_{\beta} \mathbf{R}_{\beta} \Sigma_2^{\textcircled{1}} \mathbf{J}_{\beta} \mathbf{R}_{\beta} u_0, \quad u_1 = \mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} \Sigma_2^{\textcircled{1}} \mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} \Sigma_2^{\textcircled{1}} u_1,$$

and are given in terms of each other by

$$u_0 = -\Sigma_2^{\textcircled{1}} u_1, \quad u_1 = -\mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} \Sigma_2^{\textcircled{1}} \mathbf{J}_{\beta} \mathbf{R}_{\beta} u_0.$$

*Proof.* To begin with, we write the given conditions  $\Pi_2 u = 0$  and  $\Pi_2 \mathbf{J}_{\beta} u = 0$  in the form

$$u = -\Sigma_2 u, \quad \mathbf{J}_{\beta} u = -\Sigma_2 \mathbf{J}_{\beta} u;$$

after that, we restrict to the interval  $I_1$ :

$$\mathbf{R}_1 u = -\Sigma_2^{\textcircled{1}} \mathbf{R}_1^{\dagger} u, \quad \mathbf{R}_1 \mathbf{J}_{\beta} u = \mathbf{J}_{\beta} \mathbf{R}_{\beta}^{\dagger} u = -\Sigma_2^{\textcircled{1}} \mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} u = -\Sigma_2^{\textcircled{1}} \mathbf{J}_{\beta} \mathbf{R}_{\beta} u.$$

We then simplify the second condition a little by applying  $\mathbf{J}_{\beta}$  to both sides:

$$(9.4.3) \quad \mathbf{R}_1 u = -\Sigma_2^{\textcircled{1}} \mathbf{R}_1^{\dagger} u, \quad \mathbf{R}_{\beta}^{\dagger} u = -\mathbf{J}_{\beta} \Sigma_2^{\textcircled{1}} \mathbf{J}_{\beta} \mathbf{R}_{\beta} u.$$

By combining these two identities in two separate ways, we find that

$$(9.4.4) \quad \mathbf{R}_1 u = \Sigma_2^{\textcircled{1}} \mathbf{J}_{\beta} \mathbf{R}_{\beta} \Sigma_2^{\textcircled{1}} \mathbf{J}_{\beta} \mathbf{R}_{\beta} u, \quad \mathbf{R}_{\beta}^{\dagger} u = \mathbf{J}_{\beta} \Sigma_2^{\textcircled{1}} \mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} \Sigma_2^{\textcircled{1}} \mathbf{R}_1^{\dagger} u.$$

The assertions now follow, if we use (9.4.1) and (9.4.2).  $\square$

**9.5. Two subtransfer operators on spaces of distributions.** As usual, we assume that  $0 < \beta \leq 1$ , and consider the operators

$$(9.5.1) \quad \mathbf{T}_{\beta} := \Sigma_2^{\textcircled{1}} \mathbf{J}_{\beta} \mathbf{R}_{\beta} : \mathfrak{U}(I_1) \rightarrow \mathfrak{U}(I_1),$$

and

$$(9.5.2) \quad \mathbf{V}_{\beta} := \mathbf{R}_1^{\dagger} \mathbf{J}_{\beta} \Sigma_2^{\textcircled{1}} : \mathfrak{U}(\mathbb{R} \setminus \bar{I}_1) \rightarrow \mathfrak{U}(\mathbb{R} \setminus \bar{I}_1).$$

These operators are extensions to the respective space of distributions of standard subtransfer operators. We met e.g.  $\mathbf{T}_{\beta}$  back in Subsection 1.2. Indeed, if  $u \in L^1(I_1)$  and  $v \in L^1(\mathbb{R} \setminus \bar{I}_1)$ , then

$$(9.5.3) \quad \mathbf{T}_{\beta} u(x) = \sum_{j \in \mathbb{Z}^{\times}} \frac{\beta}{(x+2j)^2} u\left(-\frac{\beta}{x+2j}\right), \quad x \in I_1,$$

and

$$(9.5.4) \quad \mathbf{V}_{\beta} v(x) = \frac{\beta}{x^2} \sum_{j \in \mathbb{Z}^{\times}} v\left(-\frac{\beta}{x} + 2j\right), \quad x \in \mathbb{R} \setminus \bar{I}_1.$$

In terms of these two subtransfer operators, the formulation of Proposition 9.4.1 simplifies pleasantly.

**Proposition 9.5.1.** Fix  $0 < \beta \leq 1$ . Suppose that for  $u \in \mathfrak{L}(\mathbb{R})$  we have  $\Pi_2 u = 0$  and  $\Pi_2 J_\beta u = 0$ . Then the restrictions  $u_0 := \mathbf{R}_1 u \in \mathfrak{L}(I_1)$  and  $u_1 := \mathbf{R}_1^\dagger u \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$  satisfy

$$u_0 = \mathbf{T}_\beta^2 u_0, \quad u_1 = \mathbf{V}_\beta^2 u_1, \quad u_0 = -\Sigma_2^\oplus u_1, \quad u_1 = -\mathbf{R}_1^\dagger J_\beta \mathbf{T}_\beta u_0.$$

*Proof.* The proof is immediate from the definitions of  $\mathbf{T}_\beta$  and  $\mathbf{V}_\beta$ .  $\square$

**Proposition 9.5.2.** Suppose that for  $u \in \mathfrak{L}(\mathbb{R})$  we have that the two restrictions vanish, i.e.,  $\mathbf{R}_1 u = 0$  and  $\mathbf{R}_1^\dagger u = 0$  as elements of  $\mathfrak{L}(I_1)$  and  $\mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$ , respectively. Then  $u = 0$ .

*Proof.* The assumption implies that the “valeur au point” function  $\text{vp}[u]$  vanishes on  $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ . But then  $\text{vp}[u]$  vanishes a.e., so that by Kolmogorov’s Proposition 4.4.2, the claim  $u = 0$  follows.  $\square$

**Remark 9.5.3.** Fix  $0 < \beta \leq 1$ . Suppose that we are given a distribution  $u_0 \in \mathfrak{L}(I_1)$  which is a fixed point for the subtransfer operator:  $\mathbf{T}_\beta^2 u_0 = u_0$ . Then the formula

$$u_1 := -\mathbf{R}_1^\dagger J_\beta \mathbf{T}_\beta u_0$$

defines a distribution  $u_1 \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$ . We quickly show that  $u_1 = \mathbf{V}_\beta^2 u_1$  and  $u_0 = -\Sigma_2^\oplus u_1$ , so that all the conditions of Proposition 9.5.1 are indeed accounted for. This means that all the solutions pairs  $(u_0, u_1)$  may be parametrized by the distribution  $u_0$  alone.

**9.6. The subtransfer operator  $\mathbf{T}_\beta$  acting on “valeur au point” functions.** The subtransfer operators  $\mathbf{T}_\beta$  and  $\mathbf{V}_\beta$  are defined on distributions, but the formulas (9.5.3) and (9.5.4) often make sense pointwise in the almost everywhere sense for functions which are not summable on the respective interval. In the sequel, we focus on  $\mathbf{T}_\beta$ ; the case of  $\mathbf{V}_\beta$  is analogous. The question appears whether for a given distribution  $u \in \mathfrak{L}(I_1)$ , with “valeur au point” function  $\text{vp}[u] \in L^{1,\infty}(I_1)$ , the action of  $\mathbf{T}_\beta$  on  $\text{vp}[u]$  by formula (9.5.3) when it converges a.e. has the same result as taking  $\text{vp}[\mathbf{T}_\beta u]$ . To analyze this, we need the finite sum operators ( $N = 2, 3, 4, \dots$ )

$$(9.6.1) \quad \mathbf{T}_\beta^{[N]} u(x) = \sum_{j \in \mathbb{Z}^\times: |j| \leq N} \frac{\beta}{(x+2j)^2} u\left(-\frac{\beta}{x+2j}\right), \quad x \in I_1.$$

This finite sum operator naturally acts both on the distribution  $u$  and on its “valeur au point” function  $\text{vp}[u]$ . As for the distributional interpretation, it is more properly understood as

$$(9.6.2) \quad \mathbf{T}_\beta^{[N]} := \Sigma_{2,N}^\oplus J_\beta \mathbf{R}_\beta,$$

where

$$\Sigma_{2,N}^\oplus : \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1) \rightarrow \mathfrak{L}(I_1)$$

is defined in the same fashion as  $\Sigma_2^\oplus$  based on the operator

$$\Sigma_{2,N} V(x) := \sum_{j \in \mathbb{Z}^\times: |j| \leq N} V(x+2j),$$

which maps  $\mathfrak{L}(\mathbb{R}) \rightarrow \mathfrak{L}(\mathbb{R})$ . Whether we apply the operator “valeur au point” before or after  $\mathbf{T}_\beta^{[N]}$  does not influence the result:

**Proposition 9.6.1.** For  $u \in \mathfrak{L}(I_1)$ , we have that

$$\text{vp}[\mathbf{T}_\beta^{[N]} u](x) = \mathbf{T}_\beta^{[N]} \text{vp}[u](x)$$

almost everywhere on the interval  $I_1$ .

*Proof.* Since the sum defining  $\mathbf{T}_\beta^{[N]} u$  is finite, it suffices to handle a single term. This amounts to showing that

$$\text{vp}\left[\frac{\beta}{(x+2j)^2} u\left(-\frac{\beta}{x+2j}\right)\right] = \frac{\beta}{(x+2j)^2} \text{vp}[u]\left(-\frac{\beta}{x+2j}\right)$$

holds almost everywhere on  $I_1$ , which is elementary.  $\square$

We can now show that  $\mathbf{T}_\beta^{[N]}u$  approximates  $\mathbf{T}_\beta u$  as  $N \rightarrow +\infty$  in terms of the “valeur au point”.

**Proposition 9.6.2.** *For  $u \in \mathfrak{L}(I_1)$ , we have that  $\mathbf{T}_\beta^{[N]}(\text{vp}[u]) \rightarrow \text{vp}[\mathbf{T}_\beta u]$  as  $N \rightarrow +\infty$  in the quasinorm of  $L^{1,\infty}(I_1)$ .*

*Proof.* We use the factorization (9.6.2), which says that  $\mathbf{T}_\beta^{[N]} = \Sigma_{2,N}^{(1)} \mathbf{J}_\beta \mathbf{R}_\beta$ . For  $v \in \mathfrak{L}(\mathbb{R} \setminus I_1)$ , we have the convergence  $\Sigma_{2,N}^{(1)} v \rightarrow \Sigma_2^{(1)} v$  in  $\mathfrak{L}(I_1)$  as  $N \rightarrow +\infty$  (cf. the proof of Proposition 6.2.1), which leads to  $\mathbf{T}_\beta^{[N]}u \rightarrow \mathbf{T}_\beta u$  in  $\mathfrak{L}(I_1)$  as  $N \rightarrow +\infty$ , for fixed  $u \in \mathfrak{L}(I_1)$ . The asserted convergence now follows from a combination of Proposition 9.6.2 with the weak-type estimate (Proposition 4.4.2).  $\square$

The Hilbert transform  $\mathbf{H}$  maps  $L_0^1(\mathbb{R}) \rightarrow \mathbf{H}L_0^1(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R})$ , and the restriction  $\mathbf{R}_1$  maps  $\mathfrak{L}(\mathbb{R}) \rightarrow \mathfrak{L}(I_1)$ , so that  $\mathbf{R}_1 \mathbf{H}$  maps  $L_0^1(\mathbb{R}) \rightarrow \mathfrak{L}(I_1)$ . By considering also the function  $P_1(t) = \pi^{-1}(1+t^2)^{-1}$ , which is in  $L^1(\mathbb{R})$  but not in  $L_0^1(\mathbb{R})$ , we realize that  $\mathbf{R}_1 \mathbf{H}$  maps  $L^1(\mathbb{R})$  into  $\mathfrak{L}(I_1)$ . We formalize this as a lemma.

**Lemma 9.6.3.** *The operator  $\mathbf{R}_1 \mathbf{H}$  maps  $L^1(\mathbb{R})$  into  $\mathfrak{L}(I_1)$ .*

**9.7. Norm expansion of the transfer operator on  $\mathfrak{L}(I_1)$ .** We now supply the proof of Theorem 1.2.1.

*Proof of Theorem 1.2.1.* Since  $\mathbf{T}_\beta = \Sigma_2^{(1)} \mathbf{J}_\beta \mathbf{R}_\beta$ , and  $\mathbf{R}_\beta$  maps  $\mathfrak{L}(I_1)$  into  $\mathfrak{L}(I_\beta)$  boundedly (more or less as a matter of definition), the boundedness of  $\mathbf{T}_\beta$  is a consequence of Propositions 9.2.1 and 9.3.3.

We turn to the assertion that the norm of  $\mathbf{T}_\beta$  exceeds 1 as an operator on  $\mathfrak{L}(I_1)$ . We recall that the norm on the space  $\mathfrak{L}(I_1)$  is induced as a quotient norm based on (1.2.8). It is straightforward to identify the dual space of  $\mathfrak{L}(\mathbb{R})$  with  $H_\otimes^\infty(\mathbb{R})$ , where the norm on  $H_\otimes^\infty(\mathbb{R})$  that is dual to (1.2.8) is given by

$$\|g\|_\otimes := \max\left(\|g\|_{L^\infty(\mathbb{R})}, \inf_{c \in \mathbb{C}} \|\tilde{\mathbf{H}}g + c\|_{L^\infty(\mathbb{R})}\right).$$

In the same fashion, the dual space of  $\mathfrak{L}(I_1)$  is identified with

$$H_\otimes^\infty(I_1) = \{g \in H_\otimes^\infty(\mathbb{R}) : \text{supp } g \subset I_1\},$$

and the corresponding norm on  $H_\otimes^\infty(I_1)$  is  $\|\cdot\|_\otimes$ . Now, by general Functional Analysis, we know that  $\|\mathbf{T}_1\| = \|\mathbf{T}_1^*\|$ , where  $\mathbf{T}_1^* = \mathbf{R}_\beta^* \mathbf{J}_\beta^* (\Sigma_2^{(1)})^* : H_\otimes^\infty(I_1) \rightarrow H_\otimes^\infty(I_1)$  and the space  $H_\otimes^\infty(I_1)$  is endowed with the norm  $\|\cdot\|_\otimes$ . Moreover, in view of the standard properties of the involution  $\mathbf{J}_\beta^*$ , it maps  $H_\otimes^\infty(\mathbb{R} \setminus I_1) \rightarrow H_\otimes^\infty(I_\beta)$  isometrically. In addition,  $\mathbf{R}_\beta^*$  is just the canonical injection  $H_\otimes^\infty(I_\beta) \rightarrow H_\otimes^\infty(I_1)$ , which is isometric as well. In conclusion, we see that  $\|\mathbf{T}_1^*\| = \|(\Sigma_2^{(1)})^*\|$ , where  $(\Sigma_2^{(1)})^*$  maps  $H_\otimes^\infty(I_1) \rightarrow H_\otimes^\infty(\mathbb{R} \setminus \bar{I}_1)$  and both spaces are endowed with the norm  $\|\cdot\|_\otimes$ . Here,  $H_\otimes^\infty(\mathbb{R} \setminus \bar{I}_1)$  denotes the subspace

$$H_\otimes^\infty(\mathbb{R} \setminus I_1) = \{g \in H_\otimes^\infty(\mathbb{R}) : \text{supp } g \subset \mathbb{R} \setminus I_1\}.$$

We proceed to show that  $\|(\Sigma_2^{(1)})^*\| > 1$ . It should be mentioned at some point that the space  $H_\otimes^\infty(I_1)$  may be identified with  $H_0^\infty(\mathbb{C} \setminus \bar{I}_1)$ , the space of bounded holomorphic functions in the slit plane  $\mathbb{C} \setminus I_1$  which also vanish at infinity. The identification is via the Cauchy transform, it is an isomorphism but it is not isometric; actually, arguably, the supremum norm on  $\mathbb{C} \setminus I_1$  might be more natural than the norm on  $H_\otimes^\infty(I_1)$  coming from the chosen norm (1.2.8) on  $\mathfrak{L}(I_1)$ . For  $0 < \gamma \leq 1$ , let us consider the function

$$G_\gamma(z) = -z^2 + z(z + \gamma) \sqrt{\frac{z - \gamma}{z + \gamma}} + \frac{\gamma^2}{2},$$

where the square root is given by the principal branch of the argument in  $\mathbb{C} \setminus \bar{\mathbb{R}}_-$ . Then  $G_\gamma \in H_0^\infty(\mathbb{C} \setminus \bar{I}_1)$ , and the corresponding element of  $H_\otimes^\infty(I_1)$  is  $g_\gamma(x) := x \sqrt{\gamma^2 - x^2} 1_{I_\gamma}(x)$ , which is

odd, with Hilbert transform

$$\mathbf{H}g_\gamma(x) = x^2 - \frac{\gamma^2}{2} - 1_{\mathbb{R} \setminus I_\gamma}(x)|x| \sqrt{x^2 - \gamma^2},$$

which is even. Both  $g_\gamma$  and  $\mathbf{H}g_\gamma$  are Hölder continuous, with  $\|g_\gamma\|_{L^\infty(\mathbb{R})} = \frac{1}{2}\gamma^2$  and

$$\inf_{c \in \mathbb{C}} \|\tilde{\mathbf{H}}g_\gamma + c\|_{L^\infty(\mathbb{R})} = \inf_{c_0 \in \mathbb{C}} \|\mathbf{H}g_\gamma + c_0\|_{L^\infty(\mathbb{R})} = \|\mathbf{H}g_\gamma\|_{L^\infty(\mathbb{R})} = \frac{\gamma^2}{2},$$

which we see from a calculation of the range of the function  $\mathbf{H}g_\gamma$ , which equals the interval  $[-\frac{1}{2}\gamma^2, \frac{1}{2}\gamma^2]$ . This gives that  $\|g_\gamma\|_{\otimes} = \frac{1}{2}\gamma^2$ . We proceed to estimate the norm  $\|(\Sigma_2^{\textcircled{1}})^*g_\gamma\|_{\otimes}$  from below. From the definition of the operator  $\Sigma_2^{\textcircled{1}}$ , we see that

$$(\Sigma_2^{\textcircled{1}})^*g_\gamma(x) = \sum_{j \in \mathbb{Z}^x} g_\gamma(x + 2j), \quad x \in \mathbb{R} \setminus \bar{I}_1,$$

and the corresponding Hilbert transform is

$$\mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(x) = \sum_{j \in \mathbb{Z}^x} \mathbf{H}g_\gamma(x + 2j) = \sum_{j=1}^{+\infty} (\mathbf{H}g_\gamma(x + 2j) + \mathbf{H}g_\gamma(x - 2j)), \quad x \in \mathbb{R}.$$

In the sum in the middle it is important to consider symmetric partial sums, which is reflected in the rightmost expression. As the sum defining  $(\Sigma_2^{\textcircled{1}})^*g_\gamma(x)$  has at most one nonzero term for each given  $x \in \mathbb{R}$ , we see that  $\|(\Sigma_2^{\textcircled{1}})^*g_\gamma\|_{L^\infty(\mathbb{R} \setminus I_1)} = \frac{1}{2}\gamma^2$ . In order to obtain the norm  $\|(\Sigma_2^{\textcircled{1}})^*g_\gamma\|_{\otimes}$ , we proceed to evaluate

$$\inf_{c_0 \in \mathbb{C}} \|\mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(x) - c_0\|_{L^\infty(\mathbb{R})}.$$

Since the functions involved are Hölder continuous and real-valued, we realize that if we may find two points  $x_1, x_2 \in \mathbb{R}$  with

$$(9.7.1) \quad \mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(x_1) - \mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(x_2) > \gamma^2,$$

then it would follow that

$$\inf_{c \in \mathbb{C}} \|\mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(x) - c\|_{L^\infty(\mathbb{R})} > \frac{\gamma^2}{2},$$

and as a consequence,  $\|\Sigma_2^{\textcircled{1}}\| = \|(\Sigma_2^{\textcircled{1}})^*\| > 1$ , as claimed. We will restrict our attention to values of  $\gamma$  that are close to 0. Taylor's formula applied to the square root function shows that

$$\mathbf{H}g_\gamma(x) = \frac{\gamma^4}{8x^2} + O\left(\frac{\gamma^6}{x^4}\right)$$

uniformly for  $|x| > 1$ . Since  $\mathbf{H}g$  is even, the value at the point  $x_2 := 2$  of the function  $\mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma$  then equals

$$\mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(x_2) = \mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(2) = \mathbf{H}g_\gamma(0) + \mathbf{H}g(2) + 2 \sum_{j=2}^{+\infty} \mathbf{H}g_\gamma(2j) = -\frac{\gamma^2}{2} + \frac{\pi^2 - 3}{96}\gamma^4 + O(\gamma^6),$$

while the value at  $x_1 := \gamma + 2N$  tends to the following value as  $N \rightarrow +\infty$  through the integers:

$$\begin{aligned} \lim_{N \rightarrow +\infty} \mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(\gamma + 2N) &= \mathbf{H}g_\gamma(\gamma) + \sum_{j=1}^{+\infty} (\mathbf{H}g_\gamma(\gamma + 2j) + \mathbf{H}g_\gamma(2j - \gamma)) \\ &= \frac{\gamma^2}{2} + 2 \sum_{j=1}^{+\infty} \mathbf{H}g_\gamma(2j) + O(\gamma^6) = \frac{\gamma^2}{2} + \frac{\pi^2 \gamma^4}{96} + O(\gamma^6). \end{aligned}$$

Finally, since

$$\lim_{N \rightarrow +\infty} \mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(\gamma + 2N) - \mathbf{H}(\Sigma_2^{\textcircled{1}})^*g_\gamma(2) = \gamma^2 + \frac{\gamma^4}{32} + O(\gamma^6) > \gamma^2$$

for small values of  $\gamma$ , we obtain (9.7.1) for  $x_1 = \gamma + 2N$  and  $x_2 = 2$ , provided  $\gamma$  is small and the positive integer  $N$  is large.  $\square$

**9.8. An operator identity of commutator type.** We recall that by Lemma 9.6.3, the operator  $\mathbf{R}_1\mathbf{H}$  maps  $L^1(\mathbb{R}) \rightarrow \mathfrak{L}(I_1)$ .

**Lemma 9.8.1.** Fix  $0 < \beta \leq 1$ . For  $f \in L^1(I_1)$ , extended to vanish off  $I_1$ , we have the identity

$$\mathbf{T}_\beta \mathbf{R}_1 \mathbf{H} f = \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta f + \mathbf{T}_\beta \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta f - \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta f,$$

as elements of the space  $\mathfrak{L}(I_1)$ .

*Proof.* In line with the presentation in the introduction, in particular, (1.2.2), we show that the claimed equality holds for  $f = \delta_\xi$ , i.e.,

$$(9.8.1) \quad \mathbf{T}_\beta \mathbf{R}_1 \mathbf{H}(\delta_\xi - \mathbf{J}_\beta \mathcal{T}_\beta \delta_\xi) = \mathbf{R}_1 \mathbf{H}(\mathcal{T}_\beta \delta_\xi - \mathbf{J}_\beta \delta_\xi)$$

holds, for almost every  $\xi \in I_1$ . The equality then holds for all  $f \in L^1(I_1)$  by “averaging”, as in (1.2.2). The canonical extension of the involution  $\mathbf{J}_\beta$  and the transfer operator  $\mathcal{T}_\beta$  to such point masses  $\delta_\xi$  reads:

$$(9.8.2) \quad \mathbf{J}_\beta \delta_\xi = \delta_{-\beta/\xi}, \quad \mathcal{T}_\beta \delta_\xi = \delta_{\{-\beta/\xi\}_2},$$

where, as in Subsection 2.1, the expression  $\{t\}_2$  stands for the real number in the interval  $]-1, 1]$  with the property that  $t - \{t\}_2 \in 2\mathbb{Z}$ . It follows that

$$(9.8.3) \quad \mathcal{T}_\beta \delta_\xi - \mathbf{J}_\beta \delta_\xi = \delta_{\{-\beta/\xi\}_2} - \delta_{-\beta/\xi}, \quad \mathbf{J}_\beta \mathcal{T}_\beta \delta_\xi = \delta_{-\beta/\{-\beta/\xi\}_2},$$

so that for  $\xi \in I_1 \setminus \bar{I}_\beta$ ,

$$\delta_\xi - \mathbf{J}_\beta \mathcal{T}_\beta \delta_\xi = 0 \quad \text{and} \quad \mathcal{T}_\beta \delta_\xi - \mathbf{J}_\beta \delta_\xi = 0.$$

It follows that for  $\xi \in I_1 \setminus \bar{I}_\beta$ , both the left-hand and the right-hand sides of the claimed equality (9.8.1) vanish, and the equality is trivially true. It remains to consider  $\xi \in \bar{I}_\beta$ . For  $\eta \in \mathbb{R}$ , the canonical extension of the Hilbert transform to a Dirac point mass at  $\eta$  is

$$\mathbf{H}\delta_\eta = \frac{1}{\pi} \text{pv} \frac{1}{x - \eta},$$

and we calculate that for two points  $\eta, \eta' \in \mathbb{R}^\times$ ,

$$\mathbf{T}_\beta \mathbf{R}_1 \mathbf{H}(\delta_\eta - \delta_{\eta'}) = \frac{1}{\pi} \text{pv} \sum_{j \in \mathbb{Z}^\times} \left( \frac{1}{x + 2j + \frac{\beta}{\eta}} - \frac{1}{x + 2j + \frac{\beta}{\eta'}} \right) \quad \text{on } I_1;$$

here, we may observe that the *principal value* interpretation is only needed with respect to at most two terms of the series. A particular instance is when

$$\frac{\beta}{\eta'} = \frac{\beta}{\eta} - 2k, \quad \text{for some } k \in \mathbb{Z},$$

in which case we get telescopic cancellation:

$$\mathbf{T}_\beta \mathbf{R}_1 \mathbf{H}[\delta_\eta - \delta_{\eta'}] = \frac{1}{\pi} \text{pv} \sum_{j \in \mathbb{Z}^\times} \left( \frac{1}{x + 2j + \frac{\beta}{\eta}} - \frac{1}{x + 2(j - k) + \frac{\beta}{\eta}} \right) = \frac{1}{\pi} \text{pv} \left\{ \frac{1}{x - 2k + \frac{\beta}{\eta}} - \frac{1}{x + \frac{\beta}{\eta}} \right\}$$

on the interval  $I_1$ . We apply this to the case  $\eta := \xi \in I_1$  and  $\eta' := -\beta/\{-\beta/\xi\}_2$ , in which case  $k \in \mathbb{Z}$  is given by

$$2k = \frac{\beta}{\xi} + \{-\beta/\xi\}_2,$$

and obtain that

$$(9.8.4) \quad \mathbf{T}_\beta \mathbf{R}_1 \mathbf{H}(\delta_\xi - \delta_{-\beta/\{-\beta/\xi\}_2}) = \frac{1}{\pi} \text{pv} \left\{ \frac{1}{x - \{-\beta/\xi\}_2} - \frac{1}{x + \frac{\beta}{\xi}} \right\} \quad \text{on } I_1.$$

The natural requirements that  $\xi \neq 0$  and that  $\{-\beta/\xi\}_2 \neq 0$  excludes a countable collection of  $\xi \in I_1$ , which has Lebesgue measure 0. By (9.8.3), this is the left-hand side expression of (9.8.1), and another application (9.8.3) gives that the right-hand side expression of (9.8.1) equals

$$(9.8.5) \quad \mathbf{R}_1 \mathbf{H}(\delta_{\{-\beta/\xi\}_2} - \delta_{-\beta/\xi}) = \frac{1}{\pi} \text{pv} \left\{ \frac{1}{x - \{-\beta/\xi\}_2} - \frac{1}{x + \frac{\beta}{\xi}} \right\} \quad \text{on } I_1.$$

From equations (9.8.4) and (9.8.5), together with (9.8.3), we find that the claimed identity (9.8.1) is correct for almost every  $\xi \in I_1$ .  $\square$

**Proposition 9.8.2.** *Fix  $0 < \beta \leq 1$ . For  $f \in L^1(I_1)$ , extended to vanish off  $I_1$ , we have the identity*

$$\mathbf{T}_\beta^n \mathbf{R}_1 \mathbf{H} f = \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^n f + \sum_{j=0}^{n-1} \left\{ \mathbf{T}_\beta^{n-j} \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{j+1} f - \mathbf{T}_\beta^{n-j-1} \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^j f \right\},$$

as elements of the space  $\mathfrak{L}(I_1)$ , for  $n = 2, 3, 4, \dots$

*Proof.* We argue by induction. First, the identity actually holds for  $n = 1$ , by Lemma 9.8.1; here, the sum from  $j = 0$  to  $j = -1$  should be understood as 0.

Next, we assume that the identity is valid for  $n = k$ , and would like to show that it holds for  $n = k + 1$  as well. From the induction hypothesis, we know that

$$(9.8.6) \quad \mathbf{T}_\beta^{k+1} \mathbf{R}_1 \mathbf{H} f = \mathbf{T}_\beta \mathbf{T}_\beta^k \mathbf{R}_1 \mathbf{H} f = \mathbf{T}_\beta \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^k f + \sum_{j=0}^{k-1} \left\{ \mathbf{T}_\beta^{k-j+1} \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{j+1} f - \mathbf{T}_\beta^{k-j} \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^j f \right\}.$$

In view of Lemma 9.8.1,

$$\mathbf{T}_\beta \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^k f = \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{k+1} f + \mathbf{T}_\beta \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{k+1} f - \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^k f,$$

and applied to (9.8.6) we obtain that

$$\begin{aligned} \mathbf{T}_\beta^{k+1} \mathbf{R}_1 \mathbf{H} f &= \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{k+1} f + \mathbf{T}_\beta \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{k+1} f - \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^k f \\ &\quad + \sum_{j=0}^{k-1} \left\{ \mathbf{T}_\beta^{k-j+1} \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{j+1} f - \mathbf{T}_\beta^{k-j} \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^j f \right\} \\ &= \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{k+1} f + \sum_{j=0}^k \left\{ \mathbf{T}_\beta^{k-j+1} \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^{j+1} f - \mathbf{T}_\beta^{k-j} \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^j f \right\} \end{aligned}$$

This is the desired identity for  $n = k + 1$ , which completes the proof.  $\square$

## 10. $\mathbf{T}_\beta$ -ITERATES OF HILBERT TRANSFORMS

**10.1. Smooth Hilbert transforms.** We fix  $0 < \beta \leq 1$ . Recall that for a function  $g \in L^1(\mathbb{R})$ , its Hilbert transform is

$$(10.1.1) \quad \mathbf{H}g(x) = \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{g(t)}{x - t} dt, \quad x \in \mathbb{R}.$$

In here, we are interested in the specific case when the function  $g$  vanishes on the interval  $I_\beta$ . Then the Hilbert transform  $\mathbf{H}g$  is smooth on  $I_\beta$ , and there is no need to considering principal values when we restrict our attention to  $I_\beta$ . In terms of the involution

$$(10.1.2) \quad \mathbf{J}_\beta g(x) = \frac{\beta}{x^2} g\left(-\frac{\beta}{x}\right),$$

we see that  $\mathbf{J}_\beta g \in L^1(I_1)$  and that

$$(10.1.3) \quad \mathbf{H}g(x) = \frac{1}{\pi} \int_{\mathbb{R} \setminus I_\beta} \frac{g(t)}{x - t} dt = \frac{1}{\pi} \int_{I_1} \frac{t}{\beta + tx} \mathbf{J}_\beta g(t) dt, \quad x \in I_\beta;$$

the advantage is that we now integrate over the symmetric unit interval  $I_1$ . In terms of the kernel

$$Q_\beta(t, x) := \frac{t}{\beta + tx}$$

and the associated integral operator

$$(10.1.4) \quad \mathbf{Q}_\beta f(x) := \frac{1}{\pi} \int_{I_1} Q_\beta(t, x) f(t) dt = \frac{1}{\pi} \int_{I_1} \frac{t}{\beta + tx} f(t) dt, \quad x \in I_\beta,$$

(10.1.3) simply asserts that

$$(10.1.5) \quad \mathbf{Q}_\beta f(x) = \mathbf{HJ}_\beta f(x), \quad x \in I_\beta,$$

for  $f \in L^1(I_1)$ , extended to vanish off  $I_1$ . It is elementary to estimate that

$$(10.1.6) \quad |Q_\beta(t, x)| = \frac{|t|}{\beta + tx} \leq \frac{2\beta}{\beta^2 - x^2} = 2\kappa_\beta(x), \quad x \in I_\beta, \quad t \in \bar{I}_1,$$

where  $\kappa_\beta$  is as in (2.7.1), which yields that

$$(10.1.7) \quad |\mathbf{Q}_\beta f(x)| \leq \frac{1}{\pi} \int_{I_1} |Q_\beta(t, x) f(t)| dt \leq \frac{2}{\pi} \|f\|_{L^1(I_1)} \kappa_\beta(x), \quad x \in I_\beta.$$

In general,  $\mathbf{Q}_\beta f$  is not in  $L^1(I_\beta)$ . But at least (10.1.7) guarantees that  $\mathbf{Q}_\beta f$  is well-defined pointwise with an effective bound. We will want to consider the  $\mathbf{T}_\beta$ -iterates of the function  $\mathbf{Q}_\beta f$ . Since, as a matter of fact, the subtransfer operator  $\mathbf{T}_\beta$  only cares about the values of the function in question on the interval  $I_\beta$ , we may use the above estimate (10.1.7) together with the observation made in Subsection 2.4 to see that the  $\mathbf{T}_\beta$ -iterates of  $\mathbf{Q}_\beta f$  are well-defined pointwise. We are also able to supply an effective estimate of those iterates, which we first do for  $0 < \beta < 1$ .

**Proposition 10.1.1.** *Fix  $0 < \beta < 1$ . Suppose  $f \in L^1(I_1)$ . Then we have the estimate*

$$|\mathbf{T}_\beta^n \mathbf{Q}_\beta f(x)| \leq \frac{4\beta^{n-1}}{\pi(1-\beta)} \|f\|_{L^1(I_1)}, \quad x \in I_1, \quad n = 2, 3, 4, \dots,$$

so that  $\mathbf{T}_\beta^n \mathbf{Q}_\beta f \rightarrow 0$  geometrically as  $n \rightarrow +\infty$ , uniformly on the interval  $I_1$ .

*Proof.* As observed above, pretty much by definition,  $\mathbf{T}_\beta g$  is only concerned with the behavior of  $g$  on the interval  $I_\beta$ . It follows from the positivity of the operator  $\mathbf{T}_\beta$  that

$$(10.1.8) \quad |\mathbf{T}_\beta \mathbf{Q}_\beta f(x)| \leq \frac{2}{\pi} \|f\|_{L^1(I_1)} \mathbf{T}_\beta \kappa_\beta(x) = \frac{2}{\pi} \|f\|_{L^1(I_1)} \kappa_1(x), \quad x \in I_1,$$

where in the last step, we used Lemma 2.7.1. Now, the same type of argument, based on Proposition 2.7.1, yields

$$|\mathbf{T}_\beta^n \mathbf{Q}_\beta f(x)| \leq \frac{2}{\pi} \|f\|_{L^1(I_1)} \mathbf{T}_\beta^{n-1} \kappa_1(x) \leq \frac{4\beta^{n-1}}{\pi(1-\beta)} \|f\|_{L^1(I_1)}, \quad x \in I_1,$$

as claimed.  $\square$

For  $\beta = 1$ , the situation is slightly more delicate.

**Proposition 10.1.2.** *Fix  $\beta = 1$ . Suppose  $f \in L^1(I_1)$ . We then have the estimate*

$$|\mathbf{T}_1^n \mathbf{Q}_1 f(x)| \leq \frac{2}{\pi} (1-x^2)^{-1} \|f\|_{L^1(I_1)}, \quad x \in I_1, \quad n = 1, 2, 3, \dots,$$

and in addition,  $\mathbf{T}_1^n \mathbf{Q}_1 f(x) \rightarrow 0$  as  $n \rightarrow +\infty$ , uniformly on compact subsets of  $I_1$ .



*Proof.* The derivation of (10.1.8) applies also in the case  $\beta = 1$ , so that

$$(10.1.9) \quad |\mathbf{T}_1 \mathbf{Q}_1 f(x)| \leq \frac{2}{\pi} \|f\|_{L^1(I_1)} \mathbf{T} \kappa_1(x) = \frac{2}{\pi} \|f\|_{L^1(I_1)} \kappa_1(x), \quad x \in I_1,$$

which is the claimed estimate for  $n = 1$ . For  $n > 1$ , we use the positivity of  $\mathbf{T}_1$  again, to obtain from (10.1.9) that

$$(10.1.10) \quad |\mathbf{T}_1^n \mathbf{Q}_1 f(x)| \leq \frac{2}{\pi} \|f\|_{L^1(I_1)} \mathbf{T}^{n-1} \kappa_1(x) = \frac{2}{\pi} \|f\|_{L^1(I_1)} \kappa_1(x), \quad x \in I_1,$$

which establishes the claimed estimate.

We proceed to obtain the uniform convergence to 0 locally on compact subsets of  $I_1$ . To this end, we use the representation (10.1.4) to see that

$$(10.1.11) \quad \mathbf{T}_1^n \mathbf{Q}_1 f(x) = \frac{1}{\pi} \int_{I_1} \mathbf{T}_1^n Q_1(t, \cdot)(x) f(t) dt.$$

We verify that for  $0 < a < 1$ ,

$$|Q_1(t, x)| \leq Q_1(a, x) = \frac{a}{1 + ax}, \quad t \in [0, a], \quad x \in I_1,$$

and that

$$|Q_1(t, x)| \leq -Q_1(-a, x) = \frac{a}{1 - ax}, \quad t \in [-a, 0], \quad x \in I_1.$$

As a consequence, using the positivity of  $\mathbf{T}_1$ , we may derive that

$$|\mathbf{T}_1^n Q_1(t, \cdot)(x)| \leq \mathbf{T}_1^n Q_1(a, \cdot)(x) \leq 2\kappa_1(x), \quad t \in [0, a], \quad x \in I_1,$$

and that

$$|\mathbf{T}_1^n Q_1(t, \cdot)(x)| \leq \mathbf{T}_1^n (-Q_1(-a, \cdot))(x) \leq 2\kappa_1(x), \quad t \in [-a, 0], \quad x \in I_1,$$

Next, we apply the triangle inequality to the integral (10.1.11):

$$(10.1.12) \quad |\mathbf{T}_1^n \mathbf{Q}_1 f(x)| \leq \frac{1}{\pi} \int_{I_a} |\mathbf{T}_1^n Q_1(t, \cdot)(x) f(t)| dt + \frac{1}{\pi} \int_{I_1 \setminus I_a} |\mathbf{T}_1^n Q_1(t, \cdot)(x) f(t)| dt \\ \leq \frac{1}{\pi} \mathbf{T}_1^n Q_1(a, \cdot)(x) \int_{[0, a]} |f(t)| dt + \frac{1}{\pi} \mathbf{T}_1^n (-Q_1(-a, \cdot))(x) \int_{[-a, 0]} |f(t)| dt + \frac{2}{\pi} \kappa_1(x) \int_{I_1 \setminus I_a} |f(t)| dt.$$

Note that in the last term, we used the estimate (10.1.6) with  $\beta = 1$ . By Proposition 2.3.1(vi),  $\mathbf{T}_1^n Q_1(a, \cdot) \rightarrow 0$  and  $\mathbf{T}_1^n Q_1(-a, \cdot) \rightarrow 0$  as  $n \rightarrow +\infty$  in the  $L^1$  sense on compact subintervals of  $I_1$ . It is a consequence of the regularity of the functions  $Q_1(a, \cdot)$  and  $Q_1(-a, \cdot)$  that the convergence is actually uniform on compact subintervals. By fixing  $a$  so close to 1 that the rightmost integral of (10.1.12) is as small as we like, we see that  $\mathbf{T}_1^n \mathbf{Q}_1 f \rightarrow 0$  as  $n \rightarrow +\infty$ , uniformly on compact subsets of  $I_1$ . This completes the proof.  $\square$

## 11. ASYMPTOTIC DECAY OF THE $\mathbf{T}_\beta$ -ORBIT OF A DISTRIBUTION IN $\mathfrak{L}(I_1)$ FOR $0 < \beta < 1$

**11.1. An application of asymptotic decay for  $0 < \beta < 1$ .** We now supply the argument which shows how, in the subcritical parameter regime  $\alpha\beta < 1$ , Theorem 1.3.1 follows from the asymptotic decay result Theorem 1.2.2, which is of extended ergodicity type.

*Proof of Theorem 1.3.1 for  $\alpha\beta < 1$ .* As observed right after the formulation of Theorem 1.3.1, a scaling argument allows us to reduce the redundancy and fix  $\alpha = 1$ , in which case the condition  $0 < \alpha\beta < 1$  reads  $0 < \beta < 1$ . In view of Subsections 8.1 and 8.2, it will be sufficient to show that for  $u \in \mathfrak{L}(\mathbb{R})$ ,

$$(11.1.1) \quad \Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0 \implies u = 0.$$

So, we assume that  $u \in \mathfrak{L}(\mathbb{R})$  has  $\Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0$ . Let  $u_0 := \mathbf{R}_1 u \in \mathfrak{L}(I_1)$  and  $u_1 := \mathbf{R}_1^\dagger u \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$  denote the restrictions of the distribution  $u$  to the symmetric interval  $I_1$  and to the complement  $\mathbb{R} \setminus \bar{I}_1$ , respectively. We will be done once we are able to show that  $u_0 = 0$ , because then  $u_1$  vanishes as well, as a result of Proposition 9.5.1:

$$u_1 = -\mathbf{R}_1^\dagger \mathbf{J}_\beta \mathbf{T}_\beta u_0 = 0.$$

Indeed, we have Proposition 9.5.2, which tells us that  $u_0 = \mathbf{R}_1 u = 0$  and  $u_1 = \mathbf{R}_1^\dagger u = 0$  together imply that  $u = 0$ .

Finally, to obtain that  $u_0 = 0$ , we observe that in addition, Proposition 9.5.1 says that  $u_0$  has the important property  $u_0 = \mathbf{T}_\beta^2 u_0$ . By iteration, then, we have  $u_0 = \mathbf{T}_\beta^{2n} u_0$  for  $n = 1, 2, 3, \dots$ , and by letting  $n \rightarrow +\infty$ , Theorem 1.2.2 tells us that  $u_0 = 0$  is the only solution in  $\mathfrak{V}(I_1)$ , which completes the proof.  $\square$

**11.2. The proof of the asymptotic decay result for  $0 < \beta < 1$ .** We now proceed with the proof of Theorem 1.2.2. Note that we have to be particularly careful because the operator  $\mathbf{T}_\beta : \mathfrak{V}(I_1) \rightarrow \mathfrak{V}(I_1)$  has norm  $> 1$ , by Theorem 1.2.1. However, it clear that it acts contractively on the subspace  $L^1(I_1)$ .

*Proof of Theorem 1.2.2.* We decompose  $u_0 = f + \mathbf{R}_1 \mathbf{H} g$ , where  $f \in L^1(I_1)$  and  $g \in L_0^1(\mathbb{R})$ , and observe that by Proposition 2.3.1(iv),

$$(11.2.1) \quad \|\mathbf{T}_\beta^N f\|_{L^1(I_1)} \rightarrow 0 \quad \text{as } N \rightarrow +\infty.$$

So the iterates  $\mathbf{T}_\beta^N f$  tend to 0 in  $L^1(I_1)$  and hence in  $L^{1,\infty}(I_1)$  as well. We turn to the  $\mathbf{T}_\beta^2$ -iterates of  $\mathbf{R}_1 \mathbf{H} g$ . First, we split

$$g = g_1 + g_2, \quad \text{where } g_1 \in L^1(I_\beta), \quad g_2 \in L^1(\mathbb{R} \setminus I_\beta);$$

here, it is tacitly assumed that the functions  $g_1, g_2$  are extended to vanish on the rest of the real line  $\mathbb{R}$ . As the operator  $\mathbf{J}_\beta$  maps  $L^1(\mathbb{R} \setminus I_\beta) \rightarrow L^1(I_1)$  isometrically, and  $\mathbf{H} g_2 = \mathbf{Q}_\beta \mathbf{J}_\beta g_2$  holds on  $I_1$  by (10.1.5), Proposition 10.1.1 gives us the pointwise estimate (we write “vp” although it is not absolutely needed)

$$(11.2.2) \quad |\text{vp}[\mathbf{T}_\beta^N \mathbf{R}_1 \mathbf{H} g_2](x)| \leq \frac{4\beta^{N-1}}{(1-\beta)\pi} \|g_2\|_{L^1(\mathbb{R})}, \quad x \in I_1.$$

In particular, the  $\mathbf{T}_\beta$ -iterates of  $\mathbf{R}_1 \mathbf{H} g_2$  tend to 0 geometrically in  $L^\infty(I_1)$ . We still need to analyze the  $\mathbf{T}_\beta^2$ -iterates of  $\mathbf{R}_1 \mathbf{H} g_1$ . We apply  $\mathbf{T}_\beta^k$  to the two sides of the identity of Proposition 9.8.2, with  $g_1$  in place of  $f$  and with  $k = 2, 3, 4, \dots$ , to obtain that

$$(11.2.3) \quad \mathbf{T}_\beta^{n+k} \mathbf{R}_1 \mathbf{H} g_1 = \mathbf{T}_\beta^k \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^n g_1 + \sum_{j=0}^{n-1} \{ \mathbf{T}_\beta^{n+k-j} \mathbf{R}_1 \mathbf{H} \mathbf{J}_\beta \mathcal{T}_\beta^{j+1} g_1 - \mathbf{T}_\beta^{n+k-j-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_\beta \mathcal{T}_\beta^j g_1 \}.$$

For  $l = 0, 1, 2, \dots$ , the function  $\mathcal{T}_\beta^l g_1$  is in  $L^1(I_1)$ , so that again by Proposition 10.1.1, since  $\mathbf{R}_1 \mathbf{H} \mathbf{J}_\beta = \mathbf{Q}_\beta$ , we have

$$(11.2.4) \quad |\text{vp}[\mathbf{T}_\beta^r \mathbf{R}_1 \mathbf{H} \mathbf{J}_\beta \mathcal{T}_\beta^l g_1](x)| \leq \frac{4\beta^{r-1}}{\pi(1-\beta)} \|g_1\|_{L^1(I_\beta)}, \quad x \in I_1, \quad r = 2, 3, 4, \dots,$$

where we use that the transfer operator  $\mathcal{T}_\beta$  acts contractively on  $L^1(I_1)$ , by Proposition 2.3.1(i). An application of the “valeur au point” estimate (11.2.4) to each term of the sum on the right-hand side of the identity (11.2.3) gives that

$$(11.2.5) \quad |\text{vp}[\mathbf{T}_\beta^{n+k} \mathbf{R}_1 \mathbf{H} g_1 - \mathbf{T}_\beta^k \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^n g_1](x)| \leq \frac{8\beta^{k-1}}{\pi(1-\beta)^2} \|g_1\|_{L^1(I_\beta)}, \quad \text{a.e. } x \in I_1.$$

Next, we split the function  $g_1$  as follows:

$$g_1 = h_{0,n} + h_{1,n}, \quad h_{0,n} \in L^1(\mathcal{E}_{\beta,n+1}), \quad h_{1,n} \in L^1(I_\beta \setminus \mathcal{E}_{\beta,n+1}),$$

where the set  $\mathcal{E}_{\beta,n+1}$  is as in (2.3.1), and with the understanding that  $h_{0,n}, h_{1,n}$  both vanish elsewhere on the real line. Next, we observe that  $\mathcal{T}_\beta^n h_{1,n} \in L^1(I_1 \setminus \bar{I}_\beta)$ . This can be seen from the

defining property of the set  $\mathcal{E}_{\beta,n+1}$  and the relation between the map  $\tau_\beta$  and the corresponding transfer operator  $\mathcal{T}_\beta$ , see (1.2.2). We then apply Proposition 10.1.1 to arrive at

$$(11.2.6) \quad \begin{aligned} |\text{vp}[\mathbf{T}_\beta^k \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^n h_{1,n}](x)| &\leq \frac{4\beta^{k-1}}{\pi(1-\beta)} \|\mathcal{T}_\beta^n h_{1,n}\|_{L^1(I_1)} \\ &\leq \frac{4\beta^{k-1}}{\pi(1-\beta)} \|h_{1,n}\|_{L^1(I_\beta)} \leq \frac{4\beta^{k-1}}{\pi(1-\beta)} \|g_1\|_{L^1(I_\beta)}, \quad x \in I_1. \end{aligned}$$

By combining (11.2.5) with the estimate (11.2.6), we obtain that

$$(11.2.7) \quad |\text{vp}[\mathbf{T}_\beta^{n+k} \mathbf{R}_1 \mathbf{H} g_1 - \mathbf{T}_\beta^k \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^n h_{0,n}](x)| \leq \frac{12\beta^{k-1}}{\pi(1-\beta)^2} \|g_1\|_{L^1(I_\beta)}, \quad \text{a.e. } x \in I_1.$$

The norm of  $h_{0,n} \in L^1(\mathcal{E}_{\beta,n+1})$  equals

$$\int_{\mathcal{E}_{\beta,n+1}} |h_{0,n}(t)| dt = \int_{\mathcal{E}_{\beta,n+1}} |g_1(t)| dt = \|1_{\mathcal{E}_{\beta,n+1}} g_1\|_{L^1(I_1)},$$

and it approaches 0 as  $n \rightarrow +\infty$ , by Proposition 2.3.1(iv). Since the transfer operator  $\mathcal{T}_\beta$  is a norm contraction on  $L^1(I_1)$ , we know that

$$\|\mathcal{T}_\beta^n h_{0,n}\|_{L^1(I_1)} \leq \|h_{0,n}\|_{L^1(I_1)} = \|1_{\mathcal{E}_{\beta,n+1}} g_1\|_{L^1(I_1)},$$

and, consequently, for fixed  $k$  we have that

$$\mathbf{T}_\beta^k \mathbf{R}_1 \mathbf{H} \mathcal{T}_\beta^n h_{0,n} \rightarrow 0 \quad \text{in } \mathfrak{L}(I_1), \quad \text{as } n \rightarrow +\infty.$$

As convergence in  $\mathfrak{L}(I_1)$  entails convergence in  $L^{1,\infty}(I_1)$  for the corresponding “valeur au point” function, we obtain from (11.2.7), by application of the  $L^{1,\infty}(I_1)$  quasinorm triangle inequality, that

$$(11.2.8) \quad \limsup_{n \rightarrow +\infty} \|\text{vp}[\mathbf{T}_\beta^{n+k} \mathbf{R}_1 \mathbf{H} g_1]\|_{L^{1,\infty}(I_1)} \leq \frac{24\beta^{k-1}}{\pi(1-\beta)^2} \|g_1\|_{L^1(I_\beta)}, \quad \text{a.e. } x \in I_1.$$

Note that the limit on the left-hand side *does not depend on the parameter  $k$* . This permits us to let  $k \rightarrow +\infty$  in a second step, and we obtain that

$$(11.2.9) \quad \lim_{N \rightarrow +\infty} \|\text{vp}[\mathbf{T}_\beta^N \mathbf{R}_1 \mathbf{H} g_1]\|_{L^{1,\infty}(I_1)} = 0.$$

Finally, gathering the terms, we obtain from (11.2.1), (11.2.2), and (11.2.9), that

$$(11.2.10) \quad \begin{aligned} \text{vp}[\mathbf{T}_\beta^N u_0] &= \text{vp}[\mathbf{T}_\beta^N (f + \mathbf{R}_1 \mathbf{H} g)] \\ &= \text{vp}[\mathbf{T}_\beta^N f] + \text{vp}[\mathbf{T}_\beta^N \mathbf{R}_1 \mathbf{H} g_1] + \text{vp}[\mathbf{T}_\beta^N \mathbf{R}_1 \mathbf{H} g_2] \rightarrow 0 \quad \text{as } N \rightarrow +\infty, \end{aligned}$$

in the quasinorm of  $L^{1,\infty}(I_1)$ , as claimed.  $\square$

*Remark 11.2.1.* One may wonder if Theorem 1.2.2 (and hence Corollary 1.2.3) would remain true if the space  $\mathfrak{L}(\mathbb{R})$  were to be replaced by the larger space  $L^{1,\infty}(I_1)$ . To look into this issue, we keep  $0 < \beta < 1$ , and consider the function

$$f(x) := \frac{1}{x - x_1} - \frac{1}{x - x_2}, \quad \text{where } x_1 := 1 + \sqrt{1 - \beta}, \quad x_2 := -1 + \sqrt{1 - \beta}.$$

Then  $x_1 x_2 = -\beta$ , so that  $\frac{\beta}{x_1} = -x_2$  and  $\frac{\beta}{x_2} = -x_1$ , and, in addition,  $\frac{\beta}{x_1} - \frac{\beta}{x_2} = x_1 - x_2 = 2$ , which leads to

$$\begin{aligned} \mathbf{T}_\beta f(x) &= \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(2j+x)^2} f\left(-\frac{\beta}{2j+x}\right) = \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(2j+x)^2} \left( \frac{1}{-\frac{\beta}{2j+x} - x_1} - \frac{1}{-\frac{\beta}{2j+x} - x_2} \right) \\ &= \sum_{j \in \mathbb{Z}^\times} \left( \frac{\beta}{(2j+x)(\beta + (2j+x)x_2)} - \frac{\beta}{(2j+x)(\beta + (2j+x)x_1)} \right) \\ &= \sum_{j \in \mathbb{Z}^\times} \left( \frac{x_1}{\beta + (2j+x)x_1} - \frac{x_2}{\beta + (2j+x)x_2} \right) = \sum_{j \in \mathbb{Z}^\times} \left( \frac{1}{2j+x+\frac{\beta}{x_1}} - \frac{1}{2j+x+\frac{\beta}{x_2}} \right) \\ &= \frac{1}{x+\frac{\beta}{x_2}} - \frac{1}{x+\frac{\beta}{x_1}} = \frac{1}{x-x_1} - \frac{1}{x-x_2} = f(x), \end{aligned}$$

by telescoping sums. The function  $f$  is a nontrivial element of  $L^{1,\infty}(I_1)$  and it is  $\mathbf{T}_\beta$ -invariant:  $\mathbf{T}_\beta f = f$ . Many other choices of the points  $x_1, x_2$  would work as well. For  $\beta = 1$ , the indicated points  $x_1, x_2$  coincide, so that  $f = 0$ , but it is enough to choose instead  $x_1 := 2 + \sqrt{3}$  and  $x_2 = -2 + \sqrt{3}$  to obtain a nontrivial function  $f$  in  $L^{1,\infty}(I_1)$  which is  $\mathbf{T}_1$ -invariant. This illustrates how Theorem 1.2.2 and Corollary 1.2.3 would utterly fail to hold if the space  $\mathfrak{L}(\mathbb{R})$  were to be replaced by  $L^{1,\infty}(I_1)$ .

## 12. THE HILBERT KERNEL AND ITS DYNAMICAL DECOMPOSITION

**12.1. Odd and even parts of the Hilbert kernel.** As in subsection 10.1, we write

$$Q_1(t, x) := \frac{t}{1+tx},$$

which is a variant of the *Hilbert kernel*. Indeed, it arises in connection with the Hilbert transform, see e.g. (10.1.5). We split the function  $Q_1$  according to odd and even parts:

$$Q_1(t, x) = Q_1^I(t, x) - Q_1^{II}(t, x), \quad Q_1^I(t, x) := \frac{t}{1-x^2t^2}, \quad Q_1^{II}(t, x) := \frac{t^2x}{1-t^2x^2}.$$

For fixed  $t \in I_1 = ]-1, 1[$ , we may calculate the action of the transfer operator  $\mathbf{T}_1$  on the function  $Q_1(t, \cdot)$  using standard trigonometric identities:

$$\begin{aligned} (12.1.1) \quad \mathbf{T}_1 Q_1(t, \cdot)(x) &= \sum_{j \in \mathbb{Z}^\times} \frac{1}{(2j+x)^2} \frac{t}{1+t(-\frac{1}{2j+x})} = \sum_{j \in \mathbb{Z}^\times} \left\{ \frac{1}{2j+x-t} - \frac{1}{2j+x} \right\} \\ &= \frac{\pi}{2} \cot\left(\frac{\pi}{2}(x-t)\right) - \frac{\pi}{2} \cot\left(\frac{\pi x}{2}\right) - \frac{t}{x(x-t)}. \end{aligned}$$

**12.2. The dynamically reduced Hilbert kernel.** Next, let  $q_1$  be the associated function

$$q_1(t, x) := (\mathbf{I} - \mathbf{T}_1)Q_1(t, \cdot)(x),$$

so that by (12.1.1),

$$(12.2.1) \quad q_1(t, x) = \frac{t}{x(x-t)} + \frac{t}{1+tx} + \frac{\pi}{2} \cot\left(\frac{\pi x}{2}\right) - \frac{\pi}{2} \cot\left(\frac{\pi}{2}(x-t)\right).$$

The function  $x \mapsto q_1(t, x)$  has removable singularities at  $x = 0$  and  $x = t$ , and poles at  $x = -2 + t$  and  $x = 2 + t$ . Therefore, the function  $x \mapsto q_1(t, x)$  has Taylor series at the origin with radius of convergence equal to  $2 - |t|$ , for  $t \in I_1$ . For fixed  $t \in I_1$ , we know that  $q_1(t, \cdot) \in L_0^1(I_1)$ , where

$$L_0^1(I_1) := \{f \in L^1(I_1) : \langle 1, f \rangle_{I_1} = 0\},$$

the reason being that

$$\langle 1, q_1(t, \cdot) \rangle_{I_1} = \langle 1, Q_1(t, \cdot) \rangle_{I_1} - \langle 1, \mathbf{T}_1 Q_1(t, \cdot) \rangle_{I_1} = \langle 1, Q_1(t, \cdot) \rangle_{I_1} - \langle \mathbf{L}_1 1, Q_1(t, \cdot) \rangle_{I_1} = 0,$$

since  $\mathbf{L}_1 1 = 1$ . We will refer to  $q_1(t, x)$  as the *dynamically reduced Hilbert kernel*.

For the endpoint parameter value  $t = 1$ , the expression for the kernel  $q_1$  is

$$(12.2.2) \quad q_1(1, x) = -\frac{1}{x} - \frac{2x}{1-x^2} + \frac{\pi}{2} \cot\left(\frac{\pi x}{2}\right) + \frac{\pi}{2} \tan\left(\frac{\pi x}{2}\right).$$

The function  $x \mapsto q_1(1, x)$  has removable singularities at  $x = 0$ ,  $x = 1$ , and  $x = -1$ , and the radius of convergence for its Taylor series at the origin equals 2. So, in particular, the function  $x \mapsto q_1(1, x)$  extends to a smooth function on the closed interval  $\bar{I}_1 = [-1, 1]$ .

**12.3. Odd and even parts of the dynamically reduced Hilbert kernel.** We split the dynamically reduced Hilbert kernel  $q_1(t, x)$  according to odd and even parts with respect to  $x$ :

$$q_1(t, x) = q_1^I(t, x) - q_1^{II}(t, x),$$

where

$$(12.3.1) \quad q_1^I(t, x) := \frac{1}{2}(q_1(t, x) + q_1(t, -x)), \quad q_1^{II}(t, x) := \frac{1}{2}(q_1(t, -x) - q_1(t, x)).$$

Obviously,  $Q_1^I(t, x)$  and  $q_1^I(t, x)$  are even functions of  $x$ , while  $Q_1^{II}(t, x)$  and  $q_1^{II}(t, x)$  are odd. By Proposition 2.5.1, the operator  $\mathbf{T}_1$  preserves even and odd symmetry, and it follows that

$$(12.3.2) \quad q_1^I(t, x) = (\mathbf{I} - \mathbf{T}_1)Q_1^I(t, \cdot)(x), \quad q_1^{II}(t, x) = (\mathbf{I} - \mathbf{T}_1)Q_1^{II}(t, \cdot)(x).$$

By inspection, the function  $q_1(1, \cdot)$  is odd, so that  $q_1^I(1, \cdot) = 0$  and  $q_1^{II}(1, \cdot) = -q_1(1, \cdot)$ . This of course corresponds to the observation that  $Q_1^I(1, x) = (1 - x^2)^{-1}$  is the density of the invariant measure. Based on (12.3.2), the standard Neumann series cancellation shows the following: for fixed  $t \in I_1$ , we have the decompositions

$$(12.3.3) \quad \begin{cases} \sum_{j=0}^{n-1} \mathbf{T}_1^j q_1^I(t, \cdot)(x) = Q_1^I(t, x) - \mathbf{T}_1^n Q_1^I(t, \cdot)(x) \\ \sum_{j=0}^{n-1} \mathbf{T}_1^j q_1^{II}(t, \cdot)(x) = Q_1^{II}(t, x) - \mathbf{T}_1^n Q_1^{II}(t, \cdot)(x). \end{cases}$$

Note that on the right-hand side of (12.3.3), the first term will tend to dominate as  $n \rightarrow +\infty$ , by Proposition 2.3.1(vi). We proceed to analyze the odd part.

**Lemma 12.3.1.** (Dynamic decomposition lemma) *For fixed  $t \in I_1$ , we have the decomposition*

$$Q_1^{II}(t, x) = \sum_{j=0}^{+\infty} \mathbf{T}_1^j q_1^{II}(t, \cdot)(x), \quad x \in I_1,$$

with uniform convergence on compact subsets of  $I_1$ , as well as norm convergence in  $L^1(I_1)$ .

*Proof.* For fixed  $t \in I_1$ , we have that  $Q_1^{II}(1, \cdot) \in L_0^1(I_1)$ , so an application of Proposition 2.3.1(v) shows that  $\mathbf{T}_1^n Q_1^{II}(t, \cdot) \rightarrow 0$  in norm in  $L^1(I_1)$  as  $n \rightarrow +\infty$ . Combined with (12.3.3), this shows that the Neumann series converges in the  $L^1(I_1)$  norm, as required. Next, the uniform convergence on compact subset follows from the  $L^1(I_1)$  convergence, combined with a comparison with the invariant measure density and a normal families argument. We leave the necessary details to the reader.  $\square$

**12.4. The fundamental estimate of the odd part of the dynamically reduced Hilbert kernel.** We will focus our attention to the odd part, which involves  $Q_1^{II}$  and  $q_1^{II}$ . Note that in view of the previous subsection, especially the formula (12.1.1), the function  $q_1^{II}$  may be expanded in the series

$$(12.4.1) \quad q_1^{II}(t, x) = \frac{t^2 x}{1 - t^2 x^2} + \sum_{j=1}^{+\infty} \left\{ \frac{2x}{4j^2 - x^2} - \frac{x}{(2j - t)^2 - x^2} - \frac{x}{(2j + t)^2 - x^2} \right\}.$$

We need effective control from above and below of the summands in Lemma 12.3.1.

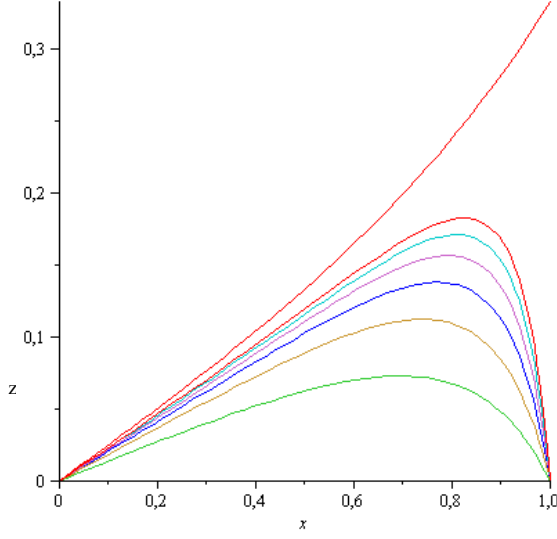


FIGURE 12.1. Illustration of the dynamic decomposition lemma (Lemma 12.3.1). The top curve is  $Q_1^{\text{II}}(t, \cdot)$ , while the curves below are the partial sums  $\sum_{j=0}^N \mathbf{T}_1^j q_1^{\text{II}}(t, \cdot)$ , with  $N = 0, 1, 2, 3, 4, 5$ . We use the parameter value  $t = 0.5$ .

Lemma 12.3.1 is basic to our analysis. We need effective control of the summands from above and below. Our result reads as follows.

**Theorem 12.4.1.** (Uniform control of summands) *For fixed  $t \in I_1$  and  $j = 0, 1, 2, 3, \dots$ , we have the following estimate:*

$$0 < \mathbf{T}_1^j q_1^{\text{II}}(t, \cdot)(x) < \mathbf{T}_1^j q_1^{\text{II}}(1, \cdot)(x), \quad x \in I_1^+.$$

We postpone the presentation of the proof until Subsections 12.5 (see Proposition 12.5.3) and 13.3 (see Corollary 13.3.3; later, the concluding steps of the proof are presented).

Of course, by the odd symmetry with respect to  $x$ , there is a corresponding estimate which holds on the left-side interval  $I_1^- := ]-1, 0[$  as well.

**12.5. Estimation from above of the odd part of the dynamically reduced Hilbert kernel.** The estimate from above in Theorem 12.4.1 will be obtained as a consequence of the following property.

**Lemma 12.5.1.** *For fixed  $t \in I_1$ , the function  $x \mapsto q_1^{\text{II}}(1, x) - q_1^{\text{II}}(t, x)$  is odd and strictly increasing on  $I_1$ .*

*Remark 12.5.2.* In view of Proposition 2.6.1,  $q_1^{\text{II}}(t, 1) = q_1^{\text{II}}(t, -1) = 0$  holds for each fixed  $t \in I_1$ . The function  $x \mapsto q_1^{\text{II}}(t, x)$  is odd, and it will be shown later that it is increasing on some interval  $I_\eta$  with  $0 < \eta < 1$ , and decreasing on the remainder set  $I_1 \setminus I_\eta$  (where the parameter  $\eta = \eta(t)$  depends on  $t$ ). However, things are a little different for  $t = 1$ . In particular, the endpoint value at  $x = \pm 1$  is different, as  $q_1^{\text{II}}(1, 1) = \frac{1}{2}$  and  $q_1^{\text{II}}(1, -1) = -\frac{1}{2}$ .

*Proof of Lemma 12.5.1.* It is obvious that the function  $x \mapsto q_1^{\text{II}}(1, x) - q_1^{\text{II}}(t, x)$  is odd. In view of the identity (12.4.1),

$$q_1(1, x) - q_1(t, x) = (\mathbf{I} - \mathbf{T}_1)[Q_1(1, \cdot) - Q_1(t, \cdot)](x) = \frac{1}{1+x} - \frac{t}{1+tx} + \sum_{j \in \mathbb{Z}^x} \left\{ \frac{1}{2j+x-t} - \frac{1}{2j+x-1} \right\},$$

and by forming the odd part with respect to the variable  $x$ , we obtain that

$$\begin{aligned} q_1^{\text{II}}(1, x) - q_1^{\text{II}}(t, x) &= (\mathbf{I} - \mathbf{T}_1)[Q_1^{\text{II}}(1, \cdot) - Q_1^{\text{II}}(t, \cdot)](x) = \frac{x}{1-x^2} - \frac{t^2x}{1-t^2x^2} \\ &\quad + \frac{1}{2} \sum_{j \in \mathbb{Z}^\times} \left\{ \frac{1}{2j-x-t} - \frac{1}{2j-x-1} - \frac{1}{2j+x-t} + \frac{1}{2j+x-1} \right\} \\ &= \frac{x}{1-x^2} - \frac{t^2x}{1-t^2x^2} + \sum_{j=1}^{+\infty} \left\{ \frac{x}{(2j-t)^2-x^2} + \frac{x}{(2j+t)^2-x^2} - \frac{x}{(2j-1)^2-x^2} - \frac{x}{(2j+1)^2-x^2} \right\}. \end{aligned}$$

In terms of the function

$$F(t, x) := \partial_x Q_1^{\text{II}}(t, x) = t^2 \frac{1+t^2x^2}{(1-t^2x^2)^2},$$

we calculate that

$$(12.5.1) \quad \partial_x(q_1^{\text{II}}(1, x) - q_1^{\text{II}}(t, x)) = F(1, x) - F(t, x) + \sum_{j=1}^{+\infty} \left( F(r_j(t), x) + F(r_j(-t), x) - F(r_j(1), x) - F(r_j(-1), x) \right),$$

where we use the notation  $r_j(t) := 1/(2j-t)$  (then  $t \mapsto r_j(t)$  is a positive and increasing function for  $j = 1, 2, 3, \dots$ ). Since the right-hand side of (12.5.1) expresses an even function of  $t$ , we may restrict our attention to  $t \in I_1^+$ . The derivative with respect to  $t$  of the function  $F(t, x)$  is

$$G(t, x) := \partial_t F(t, x) = \partial_t \partial_x Q_1^{\text{II}}(t, x) = 2t \frac{1+3t^2x^2}{(1-t^2x^2)^4},$$

and by representing differences as definite integrals of the derivative, we obtain that

$$(12.5.2) \quad \partial_x(q_1^{\text{II}}(1, x) - q_1^{\text{II}}(t, x)) = \int_t^1 G(\tau, x) d\tau + \sum_{j=1}^{+\infty} \left( \int_{r_j(-1)}^{r_j(-t)} G(\tau, x) d\tau - \int_{r_j(t)}^{r_j(1)} G(\tau, x) d\tau \right).$$

Here, we used the trivial observation that  $r_{j+1}(1) = r_j(-1)$ . Moreover, as the function  $t \mapsto G(t, x)$  is monotonically strictly increasing, we have that

$$(r_j(-t) - r_j(-1)) G(r_j(-1), x) < \int_{r_j(-1)}^{r_j(-t)} G(\tau, x) d\tau, \quad \int_{r_{j+1}(t)}^{r_j(-1)} G(\tau, x) d\tau < (r_j(-1) - r_{j+1}(t)) G(r_j(-1), x),$$

and since a trivial calculation shows that

$$r_j(-t) - 2r_j(-1) + r_{j+1}(t) = \frac{2(t-1)^2}{(2j+t)(2j+2-t)(2j+1)} > 0$$

for  $j = 1, 2, 3, \dots$  and  $t \in I_1^+$ , we obtain that

$$\int_{r_j(-1)}^{r_j(-t)} G(\tau, x) d\tau - \int_{r_{j+1}(t)}^{r_j(-1)} G(\tau, x) d\tau > 0.$$

Then, by (12.5.2), and the observation that

$$\int_t^{1/(2-t)} G(\tau, x) d\tau > 0, \quad t \in I_1^+, \quad x \in I_1,$$

it follows that the function  $x \mapsto q_1^{\text{II}}(1, x) - q_1^{\text{II}}(t, x)$  is strictly increasing, as claimed.  $\square$

We may now derive the upper bound in Theorem 12.4.1.

**Proposition 12.5.3.** *For  $j = 0, 1, 2, \dots$  and for fixed  $t \in I_1$ , the function  $\mathbf{T}_1^j[q_1^{\text{II}}(1, \cdot) - q_1^{\text{II}}(t, \cdot)]$  is odd and increasing. In particular, we have that for  $j = 0, 1, 2, \dots$  and  $t \in I_1$ ,*

$$\mathbf{T}_1^j q_1^{\text{II}}(t, \cdot)(x) < \mathbf{T}_1^j q_1^{\text{II}}(1, \cdot)(x), \quad x \in I_1^+.$$

*Proof.* This follows from a combination of Lemma 12.5.1 and Proposition 2.5.2(i).  $\square$

*Remark 12.5.4.* In general, the positivity of all the powers  $q_1^H(t, \cdot)$  on  $I_1^+ = ]0, 1[$  cannot be deduced from the simple observation that  $Q_1^H(t, \cdot)$ ,  $0 < t \leq 1$ , is odd, increasing, and positive on  $I_1^+$ . For instance, the function  $f(x) = x^3$  is odd, increasing, and positive on  $I_1^+$ . However, it can be seen that the function  $(\mathbf{I} - \mathbf{T}_1)f = f - \mathbf{T}_1 f$  changes signs on  $I_1^+$ .

### 13. POWER SERIES, HURWITZ ZETA FUNCTION, AND TOTAL POSITIVITY

**13.1. A class of power series with at most one positive zero.** The lower bound in Theorem 12.4.1 requires a more sophisticated analysis. To this end, we introduce a class of Taylor series.

Let  $\mathfrak{P}(\gamma)$  denote the class of convergent Taylor series

$$f(x) = \sum_{j=0}^{+\infty} \hat{f}(j)x^j, \quad x \in I_\gamma = ]-\gamma, \gamma[;$$

in short, we write  $f \in \mathfrak{P}(\gamma)$ . Moreover, we write  $f \in \mathfrak{P}_{\mathbb{R}}(\gamma)$  to express that the Taylor coefficients are real, that is,  $\hat{f}(j) \in \mathbb{R}$  holds for all  $j = 0, 1, 2, \dots$

**Definition 13.1.1.** Fix  $0 < \gamma < +\infty$ . If  $f \in \mathfrak{P}_{\mathbb{R}}(\gamma)$ , we write  $f \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$  to express that either

- (a)  $\hat{f}(j) \geq 0$  for all  $j = 0, 1, 2, \dots$ , or
- (b)  $\hat{f}(j) \leq 0$  for all  $j = 0, 1, 2, \dots$ , or
- (c) there exists an index  $j_0 = j_0(f) \in \mathbb{Z}_{+,0}$  such that  $\hat{f}(j) \geq 0$  for  $j \leq j_0$  while  $\hat{f}(j) \leq 0$  for  $j > j_0$ .

Under (c), assuming we have excluded the cases (a) and (b), we let  $j_0(f)$  be the *maximal* index with the property (among all the possibilities). Then  $j_0(f) \in \mathbb{Z}_{+,0}$ .

**Lemma 13.1.2.** Suppose  $f \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$ , where  $\gamma > 0$ . Then, unless  $f$  vanishes identically, there exists at most one zero of  $f$  on the interval  $]0, \gamma[$ . If such a point  $x_0 \in ]0, \gamma[$  with  $f(x_0) = 0$  exists, then  $f(x) > 0$  holds for  $0 < x < x_0$ , while  $f(x) < 0$  for  $x_0 < x < \gamma$ .

*Proof.* In accordance with the assumptions, we consider only  $f(x) \not\equiv 0$ . Note that in cases (a) or (b) of Definition 13.1.1, i.e., when  $\forall j : \hat{f}(j) \geq 0$  or  $\forall j : \hat{f}(j) \leq 0$ , then, on the interval  $[0, \gamma[$ ,  $f$  is either strictly increasing with  $f(0) = \hat{f}(0) \geq 0$ , or strictly decreasing with  $f(0) \leq 0$ , and in each case  $f$  has no zero in the interval  $]0, \gamma[$ .

It remains to deal with the case (c) of Definition 13.1.1, assuming that neither (a) nor (b) is fulfilled.

We proceed by induction on the index  $j_0(f)$ . First assume that  $j_0(f) = 0$ . In this case,  $f(x)$  is decreasing on  $[0, \gamma[$ , and it is strictly decreasing unless it is constant. If  $f(x)$  is constant, then the constant cannot be 0, and then  $f(x)$  would have no zeros at all. If  $f(x)$  is strictly decreasing instead, then it obviously can have at most one zero in the given interval, and if such a zero exists, then the values of  $f(x)$  are positive to the left of  $x_0$  and negative to the right.

Assume now that  $j_0(f) = r \geq 1$  and that the assertion of the lemma has been established for all  $f \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$  with  $j_0(f) = r - 1$ . Then the derivative  $f'(x)$  is also in the class  $\mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$ , and  $j_0(f') = r - 1$ . By the induction hypothesis, there are two possibilities:

Case (i) There is no point  $x_0$  on  $]0, \gamma[$  such that  $f'(x_0) = 0$ , and

Case (ii) There is  $x_1$ , such that  $f'(x_1) = 0$ .

In case (i),  $f'$  must have constant sign on the interval  $]0, \gamma[$ . If the sign is positive, then  $f(x)$  is increasing there, and since  $f(0) \geq 0$  the function  $f(x)$  cannot have any zeros in  $]0, \gamma[$ . If instead the sign of  $f'$  is negative, then  $f(x)$  is decreasing. If  $f(0) = 0$  the function  $f(x)$  has no zeros at all on  $]0, \gamma[$ . If  $f(0) > 0$ , then either  $f(x)$  is positive on  $]0, \gamma[$ , or it has precisely one zero  $x_0 \in ]0, \gamma[$ , in which case  $f(x)$  is positive to the left and negative to the right of  $x_0$ .

In case (ii), then by the induction hypothesis, we also know that  $f'(x) > 0$  for  $0 < x < x_1$  and that  $f'(x) < 0$  for  $x_1 < x < \gamma$ . This means that  $f(x)$  is strictly increasing on  $]0, x_1[$  and as  $f(0) \geq 0$ , it follows that  $f(x) > 0$  for  $0 < x \leq x_1$ . In the remaining interval  $x_1 < x < \gamma$ ,  $f(x)$  is decreasing,



and it is either positive throughout or has exactly one zero  $x_0 \in ]x_1, \gamma[$ , in which case  $f(x)$  is positive to the left on  $x_0$  and negative to the right. The proof is complete.  $\square$

**13.2. The Taylor coefficients of the odd part of the dynamically reduced Hilbert kernel.** We now analyze the symmetrized dynamically reduced Hilbert kernel  $q_1^{\text{II}}(t, x)$ .

**Proposition 13.2.1.** *For a fixed  $0 < t < 1$ , the function  $x \mapsto q_1^{\text{II}}(t, x)$  is odd and belongs to the class  $\mathfrak{S}_{\mathbb{R}}^{\downarrow}(\gamma)$  with  $\gamma = 2 - t$ . Indeed,  $q_1^{\text{II}}(t, \cdot)$  meets condition (c) of Definition 13.1.1.*

Before we supply the full proof of the proposition, we need to do some preparatory work. The kernel  $q_1^{\text{II}}(t, x)$  is given by (12.4.1). It is an odd function of  $x$ , and enjoys the Taylor expansion

$$(13.2.1) \quad q_1^{\text{II}}(t, x) = \sum_{j=0}^{+\infty} \kappa_j(t) x^{2j+1}, \quad x \in I_1,$$

with radius of convergence  $2 - t$ , where the coefficients can be readily calculated:

$$\kappa_j(t) := \frac{2^{-2j-2}}{(2j+1)!} \left\{ 2\psi^{(2j+1)}(1) - \psi^{(2j+1)}(1 - \frac{t}{2}) - \psi^{(2j+1)}(1 + \frac{t}{2}) \right\} + t^{2j+2}.$$

Here,

$$(13.2.2) \quad \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}, \quad \psi^{(m)}(x) = \frac{d^m}{dx^m} \frac{\Gamma'(x)}{\Gamma(x)}$$

is the *poly-Gamma function*, and  $\Gamma(x)$  is the standard *Gamma function*. A more convenient expression is obtained by direct Taylor expansion of the terms in (12.4.1):

$$\kappa_j(t) = t^{2j+2} + \sum_{k=1}^{+\infty} \left\{ \frac{2}{(2k)^{2j+2}} - \frac{1}{(2k-t)^{2j+2}} - \frac{1}{(2k+t)^{2j+2}} \right\}.$$

In terms of the *Hurwitz zeta function*

$$\zeta(s, x) := \sum_{k=0}^{+\infty} (x+k)^{-s},$$

the expression for  $\kappa_j(x)$  equals

$$\kappa_j(t) = t^{2j+2} + 2^{-2j-2} \left\{ 2\zeta(2j+2, 1) - \zeta(2j+2, 1 - \frac{t}{2}) - \zeta(2j+2, 1 + \frac{t}{2}) \right\}.$$

Moreover, since  $\zeta(s, x) = x^{-s} + \zeta(s, 1+x)$ , we may rewrite this as

$$(13.2.3) \quad \kappa_j(t) = t^{2j+2} - (2-t)^{-2j-2} + 2^{-2j-2} \left\{ 2\zeta(2j+2, 1) - \zeta(2j+2, 2 - \frac{t}{2}) - \zeta(2j+2, 1 + \frac{t}{2}) \right\}.$$

We need the following result.

**Lemma 13.2.2.** *For fixed  $\tau$  with  $0 < \tau \leq \frac{1}{2}$ , the function*

$$\Lambda_\tau(s) := (1-\tau)^s \left\{ 2\zeta(s, 1) - \zeta(s, 2-\tau) - \zeta(s, 1+\tau) \right\}$$

*is positive and strictly decreasing on the interval  $[3, +\infty[$ , with limit  $\lim_{s \rightarrow +\infty} \Lambda_\tau(s) = 0$ .*

*Proof.* We will keep the variables  $\tau$  and  $s$  confined to the indicated intervals  $0 < \tau \leq \frac{1}{2}$  and  $3 \leq s < +\infty$ .

By comparing term by term in the Hurwitz zeta sum, we see that the function  $\Lambda_\tau(s)$  is positive. Moreover, as  $s \rightarrow +\infty$ , the first term  $x^{-s}$  becomes dominant in the Hurwitz zeta series  $\zeta(s, x)$ , and we obtain that  $\Lambda_\tau(s)$  has the indicated limit.

Next, we split the function in the following way:

$$(13.2.4) \quad \Lambda_\tau(s) = \lambda_\tau(s) + R_\tau(s), \quad \lambda_\tau(s) := 2(1-\tau)^s - \left( \frac{1-\tau}{1+\tau} \right)^s$$

where  $R_\tau(s)$  is given by

$$R_\tau(s) := (1-\tau)^s \left\{ 2\zeta(s, 2) - \zeta(s, 2-\tau) - \zeta(s, 2+\tau) \right\}.$$

In the identity (13.2.4), we should think of  $\lambda_\tau(s)$  as the main term and of  $R_\tau(s)$  as the remainder. Clearly, we see that  $\lambda_\tau(s) > 0$ , and that  $\lambda_\tau(s)$  is decreasing in  $s$ :

$$(13.2.5) \quad \lambda'_\tau(s) = 2(1-\tau)^s \log(1-\tau) - \left(\frac{1-\tau}{1+\tau}\right)^s \log \frac{1-\tau}{1+\tau} \\ = (1-\tau)^s \left\{ \log(1-\tau^2) - [1 - (1+\tau)^{-s}] \log \frac{1+\tau}{1-\tau} \right\} < 0.$$

Moreover, by direct calculation

$$\partial_s \frac{\lambda'_\tau(s)}{(1-\tau)^s} = -(1+\tau)^{-s} \{\log(1+\tau)\} \log \frac{1+\tau}{1-\tau} < 0,$$

so that by (13.2.5), we have that

$$(13.2.6) \quad \lambda'_\tau(s) \leq (1-\tau)^s \frac{\lambda'_\tau(3)}{(1-\tau)^3} = (1-\tau)^s \left\{ \log(1-\tau^2) - [1 - (1+\tau)^{-3}] \log \frac{1+\tau}{1-\tau} \right\} \\ = (1-\tau)^s \left\{ \log(1-\tau^2) - \frac{\tau(3+3\tau+\tau^2)}{(1+\tau)^3} \log \frac{1+\tau}{1-\tau} \right\} \\ \leq -\tau^2(1-\tau)^s \left\{ 1 + \frac{6+6\tau+2\tau^2}{(1+\tau)^4} \right\} \leq -\frac{233}{81} \tau^2(1-\tau)^s,$$

by an elementary estimate of the logarithm function, and by using our constraints on  $s$  and  $\tau$ .

We proceed to estimate the remainder term. Since for positive  $\tau$  and a  $C^2$ -smooth function  $f$ , we have that

$$f(\tau) + f(-\tau) - 2f(0) = \int_{-\tau}^{\tau} (\tau - |\theta|) f''(\theta) d\theta,$$

and, in particular,

$$\zeta(s, 2-\tau) + \zeta(s, 2+\tau) - 2\zeta(s, 2) = \int_{-\tau}^{\tau} (\tau - |\theta|) \partial_\theta^2 \zeta(s, 2+\theta) d\theta = s(s+1) \int_{-\tau}^{\tau} (\tau - |\theta|) \zeta(s+2, 2+\theta) d\theta,$$

so that, as a consequence,

$$R_\tau(s) = -s(s+1)(1-\tau)^s \int_{-\tau}^{\tau} (\tau - |\theta|) \zeta(s+2, 2+\theta) d\theta.$$

By direct inspection, then, we see that  $R_\tau(s) < 0$ . Moreover, by differentiating the above formula with respect to  $s$ , we obtain

$$(13.2.7) \quad R'_\tau(s) = -(2s+1)(1-\tau)^s \int_{-\tau}^{\tau} (\tau - |\theta|) \zeta(s+2, 2+\theta) d\theta \\ - s(s+1) \int_{-\tau}^{\tau} (\tau - |\theta|) \partial_s [(1-\tau)^s \zeta(s+2, 2+\theta)] d\theta \\ = \frac{2s+1}{s(s+1)} R_\tau(s) - s(s+1) \int_{-\tau}^{\tau} (\tau - |\theta|) \partial_s [(1-\tau)^s \zeta(s+2, 2+\theta)] d\theta \\ \leq -s(s+1) \int_{-\tau}^{\tau} (\tau - |\theta|) \partial_s [(1-\tau)^s \zeta(s+2, 2+\theta)] d\theta.$$

We proceed to calculate the derivative which appears in (13.2.7):

$$(13.2.8) \quad -\partial_s [(1-\tau)^s \zeta(s+2, 2+\theta)] = (1-\tau)^{-2} \sum_{k=0}^{+\infty} \left( \frac{2+\theta+k}{1-\tau} \right)^{-s-2} \log \frac{2+\theta+k}{1-\tau}.$$

To analyze this derivative properly, we need the following calculation:

$$(13.2.9) \quad \partial_T \{T^{-s-2} \log T\} = -T^{-s-3} ((s+2) \log T - 1) \leq -T^{-s-3} (5 \log T - 1) < 0, \quad \frac{3}{2} \leq T < +\infty.$$

In other words,  $T \mapsto T^{-s-2} \log T$  is decreasing on the interval  $[\frac{3}{2}, +\infty[$ . As a first application of the property (13.2.9), we apply it to the identity (13.2.8) and obtain that

$$0 \leq -\partial_s[(1-\tau)^s \zeta(s+2, 2+\theta)] \leq -\partial_s[(1-\tau)^s \zeta(s+2, 2-\tau)], \quad -\tau \leq \theta \leq \tau,$$

which we may implement into (13.2.7):

$$\begin{aligned} (13.2.10) \quad R'_\tau(s) &\leq -s(s+1) \int_{-\tau}^{\tau} (\tau - |\theta|) \partial_s[(1-\tau)^s \zeta(s+2, 2+\theta)] d\theta \\ &\leq -s(s+1) \int_{-\tau}^{\tau} (\tau - |\theta|) \partial_s[(1-\tau)^s \zeta(s+2, 2-\tau)] d\theta = -s(s+1) \tau^2 \partial_s[(1-\tau)^s \zeta(s+2, 2-\tau)]. \end{aligned}$$

Next, we implement the property (13.2.9) again, in the context of (13.2.8) with  $\theta = -\tau$ :

$$\begin{aligned} (13.2.11) \quad -\partial_s[(1-\tau)^s \zeta(s+2, 2-\tau)] &= (1-\tau)^{-2} \sum_{k=0}^{+\infty} \left( \frac{2-\tau+k}{1-\tau} \right)^{-s-2} \log \frac{2-\tau+k}{1-\tau} \\ &\leq (1-\tau)^s (2-\tau)^{-s-2} \log \frac{2-\tau}{1-\tau} + (1-\tau)^{-2} \int_0^{+\infty} \left( \frac{2-\tau+x}{1-\tau} \right)^{-s-2} \log \frac{2-\tau+x}{1-\tau} dx. \end{aligned}$$

Here, we kept the first term (with  $k=0$ ), and replaced each later term indexed by  $k$  by a corresponding integral over the adjacent interval  $[k-1, k]$ . The integral expression in (13.2.11) may be calculated explicitly:

$$\begin{aligned} (13.2.12) \quad \int_0^{+\infty} \left( \frac{2-\tau+x}{1-\tau} \right)^{-s-2} \log \frac{2-\tau+x}{1-\tau} dx \\ = (s+1)^{-2} (1-\tau)^{s+2} (2-\tau)^{-s-1} \left[ 1 + (s+1) \log \frac{2-\tau}{1-\tau} \right]. \end{aligned}$$

Finally, we put (13.2.10), (13.2.11), and (13.2.12) together, and obtain that

$$\begin{aligned} (13.2.13) \quad R'_\tau(s) &\leq -s(s+1) \tau^2 \partial_s[(1-\tau)^s \zeta(s+2, 2-\tau)] \\ &\leq \tau^2 (1-\tau)^s \left\{ s(s+1) (2-\tau)^{-s-2} \log \frac{2-\tau}{1-\tau} + \frac{s}{s+1} (2-\tau)^{-s-1} \left[ 1 + (s+1) \log \frac{2-\tau}{1-\tau} \right] \right\}. \end{aligned}$$

The expression within brackets is optimized at the right end-point  $\tau = \frac{1}{2}$ :

$$\begin{aligned} (13.2.14) \quad s(s+1) (2-\tau)^{-s-2} \log \frac{2-\tau}{1-\tau} + \frac{s}{s+1} (2-\tau)^{-s-1} \left[ 1 + (s+1) \log \frac{2-\tau}{1-\tau} \right] \\ \leq s(s+1) \left( \frac{3}{2} \right)^{-s-2} \log 3 + \frac{s}{s+1} \left( \frac{3}{2} \right)^{-s-1} \left[ 1 + (s+1) \log 3 \right] \leq \frac{27}{10}, \end{aligned}$$

where the rightmost inequality is an exercise in (one-variable) Calculus. It follows from (13.2.13) and (13.2.14) that

$$(13.2.15) \quad R'_\tau(s) \leq \frac{27}{10} \tau^2 (1-\tau)^s.$$

Finally, a combination of (13.2.6) and (13.2.15) gives the desired result:

$$\Lambda'_\tau(s) = \lambda'_\tau(s) + R'_\tau(s) \leq \left( \frac{27}{10} - \frac{233}{81} \right) \tau^2 (1-\tau)^s = -\frac{143}{810} \tau^2 (1-\tau)^s < 0.$$

The proof is complete.  $\square$

**Proposition 13.2.3.** *For fixed  $t$ ,  $0 < t < 1$ , the function  $j \mapsto (2-t)^{2j+2} \kappa_j(t)$  is strictly decreasing on  $\mathbb{Z}_+ = \{1, 2, \dots\}$ , with limit*

$$\lim_{j \rightarrow +\infty} (2-t)^{2j+2} \kappa_j(t) = -1.$$

*Proof.* In view of (13.2.3), we know that

$$(2-t)^{2j+2}\kappa_j(t) = [t(2-t)]^{2j+2} - 1 + \Lambda_{t/2}(2j+2).$$

Since  $0 < t(2-t) < 1$  holds for  $t \in I_1^+$ , the function  $j \mapsto [t(2-t)]^{2j+2}$  is decreasing, and the lemma becomes an immediate consequence of Lemma 13.2.2.  $\square$

*Proof of Proposition 13.2.1.* It is clear from the known radius of convergence for  $q_1^{\text{II}}(t, \cdot)$  that  $q_1^{\text{II}}(t, \cdot) \in \mathfrak{B}(2-t)$ . Moreover,  $q_1^{\text{II}}(t, \cdot)$  is odd, and all the Taylor coefficients (see (13.2.1)) are clearly real-valued, while the coefficient of the linear term is explicit and positive:

$$\kappa_0(t) := \frac{\pi^2}{12} + \frac{1}{t^2} - \frac{\pi^2/4}{\sin^2(\frac{\pi}{2}t)} + t^2 > 0.$$

Now, the proof of the proposition is an immediate consequence of Proposition 13.2.3.  $\square$

**13.3. Positivity of the odd part of the dynamically reduced Hilbert kernel and totally positive matrices.** The transfer operator  $\mathbf{T}_1$  can be applied to polynomials, or, more generally, convergent power series. For  $j = 0, 1, 2, \dots$ , let  $u_j$  denote the monomial  $u_j(x) := x^{2j+1}$ . The action of  $\mathbf{T}_1$  on odd power series can be analyzed in terms of the infinite matrix  $\mathbf{B} = \{b_{j,k}\}_{j,k=0}^{+\infty}$  with entries  $b_{j,k}$  given by

$$(13.3.1) \quad \mathbf{T}_1 u_j(x) = \sum_{k=0}^{+\infty} b_{j,k} u_k, \quad x \in I_1,$$

since the transfer operator  $\mathbf{T}_1$  preserves oddness.

We recall the notion of a *totally positive matrix* [16]. An infinite matrix  $\mathbf{A} = \{a_{j,k}\}_{j,k=0}^{+\infty}$  is said to be *totally positive* if all its minors are  $\geq 0$ , and *strictly totally positive* if all its minors are  $> 0$ . Here, a minor is the determinant of a square submatrix  $\{a_{j_s, k_t}\}_{s,t=1}^r$  where  $j_1 < \dots < j_r$  and  $k_1 < \dots < k_r$ . This is a much stronger property than the usual positive definiteness of a matrix, which would correspond to considering only symmetric squares.

**Proposition 13.3.1.** *The matrix  $\mathbf{B} = \{b_{j,k}\}_{j,k=0}^{+\infty}$  with coefficients given by (13.3.1) is strictly totally positive.*

*Proof.* We read off from the definition of  $\mathbf{T}_1$  that

$$\mathbf{T}_1 u_j(x) = - \sum_{n \in \mathbb{Z}^+} (x + 2n)^{-2j-3},$$

and observe that the right-hand side may be written in the form

$$\mathbf{T}_1 u_j(x) = \frac{1}{2^{2j+2}(2j+2)!} \left\{ \psi^{(2j+2)}\left(1 + \frac{x}{2}\right) - \psi^{(2j+2)}\left(1 - \frac{x}{2}\right) \right\},$$

where  $\psi^{(m)}(x)$  is the poly-Gamma function (see (13.2.2)). From this, we immediately obtain

$$b_{j,k} = \frac{\psi^{(2j+2k+3)}(1)}{2^{2j+2k+3}(2j+2)!(2k+1)!}.$$

Since strict total positivity remains unchanged as we multiply a column or a row by a positive number, the strict total positivity of the matrix  $\mathbf{B}$  is equivalent to the strict total positivity of the infinite matrix with entries  $\{c_{j+k}\}_{j,k=0}^{+\infty}$  where  $c_j := \psi^{(2j+3)}(1)$ . This is a Hankel matrix, and in view of Theorem 4.4 [16], its total positivity is equivalent to the strict positive definiteness of all the finite square matrices  $\{c_{j+l}\}_{j,l=0}^N$  and  $\{c_{j+l+1}\}_{j,l=0}^{N-1}$  for every  $N = 1, 2, 3, \dots$ . Following the digression in Section 4.6 of [16], we know that this is equivalent to having the  $c_j$  be the moments of a positive measure (the Stieltjes moment problem). However, it is known that

$$c_j = \psi^{(2j+3)}(1) = \int_0^{+\infty} t^{2j+3} \frac{e^{-t}}{1-e^{-t}} dt = \int_0^{+\infty} t^j \frac{t e^{-\sqrt{t}}}{1-e^{-\sqrt{t}}} dt,$$

which means that the  $c_j$  are indeed the moments of a positive measure. This completes the proof.  $\square$

We need to have a precise definition of the notion of counting *sign changes*, see [16].

**Definition 13.3.2.** Let  $\mathbf{a} = \{a_j\}_j$ ,  $j = 0, \dots, N$ , be a finite sequence of real numbers.

(a) The number  $S^-(\mathbf{a})$  counts the number of sign changes in the sequence with zero terms discarded. This is the number of *strong sign changes*.

(b) The number  $S^+(\mathbf{a})$  counts the maximal number of sign changes in the sequence, where zero terms are arbitrarily replaced by  $+1$  or  $-1$ . This is the number of *weak sign changes*.

Obviously, the number of weak sign changes exceeds the number of strong sign changes, i.e.,  $S^-(\mathbf{a}) \leq S^+(\mathbf{a})$ .

**Corollary 13.3.3.** Fix  $1 \leq \gamma < +\infty$ . If  $f \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$  is odd, then  $\mathbf{T}_1 f$  is odd as well, and  $\mathbf{T}_1 f \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(2 - \frac{1}{\gamma})$ .

*Proof.* Based on the explicit expression (1.2.5) for  $\mathbf{T}_1 f$ , it is a straightforward exercise in the analysis of power series to check that if  $f \in \mathfrak{S}(\gamma)$ , then  $\mathbf{T}_1 f \in \mathfrak{P}(2 - \frac{1}{\gamma})$ . Moreover, it is clear that the property of having real Taylor coefficients is preserved under  $\mathbf{T}_1$ . To finish the proof, we pick an odd  $f \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$ , and expand it in a Taylor series:

$$f(x) = \sum_{j=0}^{+\infty} \hat{f}(2j+1)u_j(x), \quad -\gamma < x < \gamma.$$

where as before  $u_j(x) = x^{2j+1}$ . Then, in view of (13.3.1),

$$(13.3.2) \quad \mathbf{T}_1 f(x) = \sum_{j=0}^{+\infty} \hat{f}(2j+1)\mathbf{T}_1 u_j(x) = \sum_{k=0}^{+\infty} \left\{ \sum_{j=k}^{+\infty} b_{j,k} \hat{f}(2j+1) \right\} u_j(x), \quad -1 < x < 1,$$

and, as noted before, the right-hand side Taylor series converges in the interval  $] -2 + \frac{1}{\gamma}, 2 - \frac{1}{\gamma} [$ . The assertion of the corollary is trivial if  $f(x) \equiv 0$ , so we may assume that  $f$  does not vanish identically. From the definition of the class  $\mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma)$ , we read off that for  $N = 1, 2, 3, \dots$ , the finite sequence  $\{\hat{f}(2j+1)\}_{j=0}^N$  has *at most one strong sign change*. Next, by Proposition 13.3.1, we may apply the Variation Diminishing Theorem for strictly totally positive matrices (see Theorem 3.3 in [16]), which asserts that the sequence  $\{F_{k,N}\}_{k=0}^N$ , where

$$(13.3.3) \quad F_{k,N} := \sum_{j=0}^N b_{j,k} \hat{f}(2j+1),$$

has *at most one weak sign change* in the index interval  $\{0, \dots, N\}$ . Moreover, if there is a weak sign change in the sequence  $\{F_{k,N}\}_{k=0}^N$ , then it is from  $\geq 0$  on the left to  $\leq 0$  on the right. More precisely, we have the following three possibilities:

- (i)  $F_{k,N} \geq 0$  for all  $k = 0, \dots, N$ , or
- (ii)  $F_{k,N} \leq 0$  for all  $k = 0, \dots, N$ , or
- (iii) there exists an index  $k_0 \in \{0, \dots, N-1\}$  such that  $F_{k,N} \geq 0$  for  $k = 0, \dots, k_0$  while  $F_{k,N} \leq 0$  for  $k = k_0 + 1, \dots, N$ .

As we let  $N \rightarrow +\infty$ , the coefficients  $F_{k,N}$  converge to

$$F_k := \sum_{j=0}^{+\infty} b_{j,k} \hat{f}(2j+1),$$

where the right-hand side is absolutely convergent because all the coefficients (except possibly a finite number of them) has the same sign. From the properties (i)–(iii), we see that the sequence  $\{F_k\}_k$  has one of the following three properties:

- (i')  $F_k \geq 0$  for all  $k = 0, 1, 2, \dots$ , or

(ii')  $F_k \leq 0$  for all  $k = 0, 1, 2, \dots$ , or  
 (iii') there exists an index  $k_0 \in \mathbb{Z}_{+,0}$  such that  $F_k \geq 0$  for  $k = 0, \dots, k_0$  while  $F_{k,N} \leq 0$  for  $k = k_0 + 1, k_0 + 2, \dots$

We remark that while it is clear that property (i) converges to (i'), and that (ii) converges to (ii'), the case (iii) is less stable and might degenerate into (i') or (ii'), as  $N \rightarrow +\infty$ . No matter which of these cases (i')–(iii') we are in, the corresponding Taylor series

$$\mathbf{T}_1 f(x) = \sum_{k=0}^{+\infty} F_k x^{2k+1}, \quad -1 < x < 1,$$

is odd and belongs to  $\mathfrak{P}_{\mathbb{R}}^{\downarrow}(2 - \frac{1}{\gamma})$ . The proof is complete.  $\square$

We now turn to the proof of Theorem 12.4.1.

*Proof of Theorem 12.4.1.* As the required estimate from above was obtained back in Proposition 12.5.3, we may concentrate on the estimate from below.

The function  $t \mapsto q_1^H(t, x)$  is clearly even, and then the iterates  $\mathbf{T}_1 q_1^H(t, \cdot)$  are also even with respect to the parameter  $t$ . So, by symmetry, it will be enough to treat the case  $0 < t < 1$ . So, we assume  $0 < t < 1$ , and observe that Proposition 13.2.1 asserts that  $q_1^H(t, \cdot)$  is odd and belongs to the class  $\mathfrak{P}_{\mathbb{R}}^{\downarrow}(2 - t)$ . Next, by applying Corollary 13.3.3 once, we have that  $\mathbf{T}_1 q_1^H(t, \cdot)$  is odd as well and belongs to  $\mathfrak{P}_{\mathbb{R}}^{\downarrow}(2 - \frac{1}{2-t})$ . Here, we note that  $1 < 2 - t < 2$  and  $1 < 2 - \frac{1}{2-t} < \frac{3}{2}$ . By applying Corollary 13.3.3 iteratively, we find more generally that  $\mathbf{T}_1^j q_1^H(t, \cdot) \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma_j(t))$ , for some  $\gamma_j(t)$  with  $\gamma_j(t) > 1$ . Now, since  $q_1^H(t, \cdot)$  is odd, we may apply repeatedly Proposition 2.6.1, to see that

$$(13.3.4) \quad \mathbf{T}_1^j q_1^H(t, \cdot)(1) = \mathbf{T}_1^{j-1} q_1^H(t, \cdot)(1) = \dots = \mathbf{T}_1 q_1^H(t, \cdot)(1) = q_1^H(t, \cdot)(1) = 0.$$

Next, by (13.3.4) and Lemma 13.1.2, we find that

$$\mathbf{T}_1^j q_1^H(t, \cdot)(x) > 0, \quad 0 < x < 1, \quad j = 1, 2, 3, \dots$$

unless the function  $\mathbf{T}_1^j q_1^H(t, \cdot) \in \mathfrak{P}_{\mathbb{R}}^{\downarrow}(\gamma_j(t))$  vanishes identically. To rule out the latter possibility, we argue as follows. If  $\mathbf{T}_1^j q_1^H(t, \cdot) = 0$ , then we would have that

$$0 = \mathbf{T}_1^j q_1^H(t, \cdot) = \mathbf{T}_1^j (\mathbf{I} - \mathbf{T}_1) Q_1^H(t, \cdot) = (\mathbf{I} - \mathbf{T}_1) \mathbf{T}_1^j Q_1^H(t, \cdot),$$

that is,  $\mathbf{T}_1^j Q_1^H(t, \cdot) \in L^1(I_1)$  would be an eigenfunction for the operator  $\mathbf{T}_1$  corresponding to the eigenvalue 1, which is only possible (in view of Proposition 10.1.2) when  $\mathbf{T}_1^j Q_1^H(t, \cdot) = 0$ . This is absurd, as  $\mathbf{T}_1$  preserves the class of odd strictly increasing functions, see Proposition 2.5.2(i). The proof is complete.  $\square$

#### 14. ASYMPTOTIC DECAY OF THE $\mathbf{T}_1$ -ORBIT OF AN ODD DISTRIBUTION IN $\mathfrak{Q}(I_1)$

**14.1. An application of asymptotic decay for  $\beta = 1$ .** We now explain how to obtain, in the critical parameter regime  $\alpha\beta = 1$ , Theorem 1.3.1 as a consequence of Theorem 1.2.4.

*Proof of Theorem 1.3.1 for  $\alpha\beta = 1$ .* As observed right after the formulation of Theorem 1.3.1, a scaling argument allows us to reduce the redundancy and fix  $\alpha = 1$ , in which case the condition  $0 < \alpha\beta = 1$  reads  $\beta = 1$ . In view of Subsections 8.1 and 8.2, it will be sufficient to show that for  $u \in \mathfrak{Q}_0(\mathbb{R})$ ,

$$(14.1.1) \quad \Pi_2 u = \Pi_2 \mathbf{J}_1 u = 0 \implies u = 0.$$

Here, we recall the notation

$$\mathfrak{Q}_0(\mathbb{R}) := L_0^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R}) \subset \mathfrak{Q}(\mathbb{R}).$$

So, we assume that  $u \in \mathfrak{L}_0(\mathbb{R})$  has  $\Pi_2 u = \Pi_2 J_1 u = 0$ . The distribution  $u$  has a decomposition  $u = f + \mathbf{H}g$ , where  $f, g \in L_0^1(\mathbb{R})$ . We write

$$(14.1.2) \quad f^I(t) = \frac{1}{2}(f(t) + f(-t)), \quad f^{II}(t) = \frac{1}{2}(f(-t) - f(t)),$$

and

$$(14.1.3) \quad g^I(t) = \frac{1}{2}(g(t) + g(-t)), \quad g^{II}(t) = \frac{1}{2}(g(-t) - g(t)),$$

so that the functions  $f^I, g^I \in L_0^1(\mathbb{R})$  are even while  $f^{II}, g^{II} \in L_0^1(\mathbb{R})$  are odd. We then put

$$u^I = f^I - \mathbf{H}g^{II}, \quad u^{II} = f^{II} - \mathbf{H}g^I,$$

so that  $u^I \in \mathfrak{L}_0(\mathbb{R})$  is an even distribution, while  $u^{II}$  is odd. This is so because the Hilbert transform is symmetry reversing, odd is mapped to even, and even to odd.

STEP I: We first prove that the implication (14.1.1) holds for odd  $u$ , that is, when  $u = -u^{II}$ :

$$(14.1.4) \quad \Pi_2 u^{II} = \Pi_2 J_1 u^{II} = 0 \iff u^{II} = 0.$$

The added arrow to the left is of course a trivial implication. Let  $u_0^{II} := \mathbf{R}_1 u^{II} \in \mathfrak{L}(I_1)$  and  $u_1^{II} := \mathbf{R}_1^\dagger u^{II} \in \mathfrak{L}(\mathbb{R} \setminus \bar{I}_1)$  denote the restrictions of the distribution  $u^{II}$  to the symmetric interval  $I_1$  and to the complement  $\mathbb{R} \setminus \bar{I}_1$ , respectively. Clearly,  $u_0^{II}$  and  $u_1^{II}$  are odd, because  $u^{II}$  is. We will be done with this step once we are able to show that  $u_0^{II} = 0$ , because then  $u_1^{II}$  vanishes as well, as a result of Proposition 9.5.1:

$$u_1^{II} = -\mathbf{R}_1^\dagger J_1 T_1 u_0^{II} = 0.$$

Indeed, we have Proposition 9.5.2, which tells us that  $u_0^{II} = \mathbf{R}_1 u^{II} = 0$  and  $u_1^{II} = \mathbf{R}_1^\dagger u^{II} = 0$  together imply that  $u^{II} = 0$ . Finally, to obtain that  $u_0^{II} = 0$ , we observe that in addition, Proposition 9.5.1 says that the odd distribution  $u_0^{II} \in \mathfrak{L}(I_1)$  has the important property  $u_0^{II} = \mathbf{T}_1^2 u_0^{II}$ . By iteration, then, we have  $u_0^{II} = \mathbf{T}_1^{2n} u_0^{II}$  for  $n = 1, 2, 3, \dots$ , and by letting  $n \rightarrow +\infty$ , we realize from Theorem 1.2.4 that  $u_0 = 0$  is the only possible solution in  $\mathfrak{L}(I_1)$ .

STEP II: We now prove, based on Step I, that the implication (14.1.1) holds for an arbitrary distribution  $u \in \mathfrak{L}_0(\mathbb{R})$ , regardless of symmetry. So, we take a distribution  $u \in \mathfrak{L}_0(\mathbb{R})$  for which  $\Pi_2 u = 0$  and  $\Pi_2 J_1 u = 0$ . We split  $u = u^I - u^{II}$  as above, and note that since the operators  $\Pi_2$  and  $J_1$  both respect odd-even symmetry,

$$0 = \Pi_2 u = \Pi_2 u^I - \Pi_2 u^{II} \quad \text{and} \quad 0 = \Pi_2 J_1 u = \Pi_2 J_1 u^I - \Pi_2 J_1 u^{II}$$

correspond to the splitting of the 0 distribution into odd-even parts inside the space

$$\mathfrak{L}_0(\mathbb{R}/2\mathbb{Z}) := L_0^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L_0^1(\mathbb{R}/2\mathbb{Z}) \subset \mathfrak{L}(\mathbb{R}/2\mathbb{Z}).$$

This means that each part must vanish separately, that is,

$$(14.1.5) \quad \Pi_2 u^I = 0, \quad \Pi_2 J_1 u^I = 0, \quad \Pi_2 u^{II} = 0, \quad \Pi_2 J_1 u^{II} = 0.$$

By Step I, we know that the implication (14.1.5) holds for the odd distribution  $u^{II}$ , so it is an immediate consequence of (14.1.5) that  $u^{II} = 0$ . We need to understand the result obtained in Step I better, and write the equivalence (14.1.4) in terms of the functions  $f^{II}$  and  $g^I$ :

$$(14.1.6) \quad f^{II} = \mathbf{H}g^I \iff \begin{cases} \Pi_2 f^{II} = \Pi_2 \mathbf{H}g^I, \\ \Pi_2 J_1 f^{II} = \Pi_2 \mathbf{H}J_1 g^I. \end{cases}$$

Next, since we know that  $\Pi_2 \mathbf{H} = \mathbf{H}_2 \Pi_2$  as operators on  $L_0^1(\mathbb{R})$ , we may rewrite (14.1.6) as

$$(14.1.7) \quad f^{II} = \mathbf{H}g^I \iff \begin{cases} \Pi_2 f^{II} = \mathbf{H}_2 \Pi_2 g^I, \\ \Pi_2 J_1 f^{II} = \mathbf{H}_2 \Pi_2 J_1 g^I. \end{cases}$$

Since we already know that  $u^{II} = 0$ , it remains to explain why  $u^I = 0$  must hold as well. The relation (14.1.5) also contains the conditions  $\Pi_2 u^I = \Pi_2 J_1 u^I = 0$ , which in terms of  $f^I$  and  $g^{II}$  amount to having

$$\begin{cases} \Pi_2 f^I = H_2 \Pi_2 g^{II}, \\ \Pi_2 J_1 f^I = H_2 \Pi_2 J_1 g^{II}. \end{cases}$$

Let us apply the periodic Hilbert transform  $H_2$  to the left-hand and right-hand sides, which is an invertible transformation on  $\mathfrak{L}_0(\mathbb{R}/2\mathbb{Z})$  with  $H_2^2 = -I$ . The result is

$$\begin{cases} H_2 \Pi_2 f^I = -\Pi_2 g^{II}, \\ H_2 \Pi_2 J_1 f^I = -\Pi_2 J_1 g^{II}. \end{cases}$$

But this places us in the setting of (14.1.7), only with  $-g^{II}$  in place of  $f^{II}$ , and  $f^I$  in place of  $g^I$ . So we get from (14.1.7) that  $-g^{II} = H f^I$ , which after application of  $H$  reads  $f^I = H g^{II}$ . This means that  $u^I = f^I - H g^{II} = 0$ , as desired. Finally, since both  $u^I$  and  $u^{II}$  vanish, we obtain  $u = u^I - u^{II} = 0$ . This proves that the implication (14.1.1) holds for every  $u \in \mathfrak{L}_0(\mathbb{R})$ , which completes the proof.  $\square$

**14.2. The proof of asymptotic decay for  $\beta = 1$ .** We now proceed with the proof of Theorem 1.2.4. As in the proof of Theorem 1.2.2, we have to be particularly careful because the operator  $T_1 : \mathfrak{L}(I_1) \rightarrow \mathfrak{L}(I_1)$  has norm  $> 1$ . However, it clearly acts contractively on  $L^1(I_1)$ .

*Proof of Theorem 1.2.4.* Since  $u_0 \in \mathfrak{L}(I_1)$ , we know that there exist functions  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$  such that  $u_0 = R_1(f + Hg)$ .

**STEP I:** We find a suitable odd extension of  $u_0$  to all of  $\mathbb{R}$ . Let the functions  $f^I, f^{II}, g^I, g^{II}$  be given by (14.1.2) and (14.1.3), and put

$$u^I = f^I - H g^{II}, \quad u^{II} = f^{II} - H g^I,$$

so that  $u^I \in \mathfrak{L}(\mathbb{R})$  is an even distribution, while  $u^{II} \in \mathfrak{L}(\mathbb{R})$  is odd. The way things are set up, we have that  $u_0 = R_1 u$ , where  $u := u^I - u^{II}$ . Since it is given that  $u_0$  is odd, we must have that  $R_1 u^I = 0$ , and that  $u_0 = -R_1 u^{II}$ . The distribution  $-u^{II}$  is odd on all of  $\mathbb{R}$ , and provides an extension of  $u_0$  beyond the interval  $I_1$ . We will focus our attention on the odd distribution  $u^{II} = f^{II} - H g^I$ , which has  $R_1 u^{II} = -u_0$ .

**STEP II:** We now argue that without loss of generality, we may require of the even function  $g^I \in L_0^1(\mathbb{R})$  that in addition

$$(14.2.1) \quad \langle 1, g^I \rangle_{I_1} = \langle 1, g^I \rangle_{\mathbb{R} \setminus I_1} = 0.$$

To this end, we consider the even function  $h^I \in L_0^1(\mathbb{R})$  given by

$$h^I(x) := 1_{I_1}(x) - x^{-2} 1_{\mathbb{R} \setminus I_1}(x), \quad x \in \mathbb{R},$$

with Hilbert transform

$$H h^I(x) = \frac{1}{\pi} \log \left| \frac{x+1}{x-1} \right| + \frac{1}{\pi x^2} \log \left| \frac{x+1}{x-1} \right| - \frac{2}{\pi x}, \quad x \in \mathbb{R},$$

and note that  $H h^I \in L_0^1(\mathbb{R})$  is odd. Now, if (14.2.1) is not fulfilled to begin with, then we consider instead the functions

$$F^{II} := f^{II} + \frac{1}{2} \langle g^I, 1 \rangle_{I_1} H h^I, \quad G^I := g^I - \frac{1}{2} \langle g^I, 1 \rangle_{I_1} h^I.$$

Indeed, we see that  $F^{II}, G^I \in L_0^1(\mathbb{R})$  where  $F^{II}$  is odd and  $G^I$  is even, that (14.2.1) holds with  $G^I$  in place of  $g^I$ , and that  $u^{II} = f^{II} - g^I = F^{II} - H G^I$ . This allows us assume that  $f^{II}, g^I \in L_0^1(\mathbb{R})$  are chosen so that (14.2.1) holds, and completes the proof of Step II.

**STEP III:** *Splitting of the functions  $f^{II}$  and  $g^I$  according to intervals.* We split  $f = f_1 + f_2$  and  $g = g_1 + g_2$ , where

$$f_1^{II} := f^{II} 1_{I_1} \in L_0^1(I_1), \quad f_2^{II} := f^{II} 1_{\mathbb{R} \setminus I_1} \in L_0^1(\mathbb{R} \setminus I_1),$$



and

$$g_1 := g1_{I_1} \in L_0^1(I_1), \quad g_2 := g1_{\mathbb{R} \setminus I_1} \in L_0^1(\mathbb{R} \setminus I_1).$$

Here, we used in fact Step II. Note that the functions  $f_1^H, f_2^H$  are odd, while  $g_1^I, g_2^I$  are even. We write  $u_0^H := \mathbf{R}_1 u_0^H$ , so that  $u_0^H = -u_0$ . We note that

$$(14.2.2) \quad u_0^H = \mathbf{R}_1(f_1^H + \mathbf{H}g_1^I) = \mathbf{R}_1(f_1^H + f_2^H + \mathbf{H}g_1^I + \mathbf{H}g_2^I) = f_1^H + \mathbf{R}_1\mathbf{H}g_1^I + \mathbf{R}_1\mathbf{H}g_2^I.$$

By applying the operator  $\mathbf{T}_1^N$  for  $N = 2, 3, 4, \dots$  to the leftmost and rightmost sides of (14.2.2), we obtain

$$\mathbf{T}_1^N u_0^H = \mathbf{T}_1^N f_1^H + \mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_1^I + \mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_2^I.$$

and after application of the “valeur au point” operation, this identity reads, for  $N = 2, 3, 4, \dots$ ,

$$(14.2.3) \quad \text{vp}[\mathbf{T}_1^N u_0^H](x) = \mathbf{T}_1^N f_1^H(x) + \text{vp}[\mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_1^I](x) + \text{vp}[\mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_2^I](x), \quad \text{a.e. } x \in I_1.$$

Next, by Propositions 2.3.1(v) and 10.1.2, we have for each fixed  $\eta, 0 < \eta < 1$ , the convergence

$$(14.2.4) \quad \mathbf{T}_1^N f_1^H \rightarrow 0 \quad \text{and} \quad 1_{I_\eta} \text{vp}[\mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_2^I] \rightarrow 0,$$

the first one in the norm of  $L^1(I_1)$  as  $N \rightarrow +\infty$ . That is, two terms on the right-hand side of (14.2.3) vanish in the limit on compact subintervals, and we are left to analyze the remaining middle term.

By rearranging the terms in the finite expansion of Proposition 9.8.2 with  $n := N$ , applied to the even function  $g_1^I \in L_0^1(I_1)$  in place of  $f$ , we obtain that

$$(14.2.5) \quad \mathbf{T}_1^N \mathbf{R}_1 \mathbf{H} g_1^I = \mathbf{R}_1 \mathbf{H} \mathbf{T}_1^N g_1^I - \mathbf{T}_1^{N-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 g_1^I + \mathbf{T}_1 \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathcal{T}_1^N g_1^I \\ + \sum_{j=1}^{N-1} (\mathbf{T}_1^2 - \mathbf{I}) \mathbf{T}_1^{N-j-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathcal{T}_1^j g_1^I,$$

Here, of course,  $\mathcal{T}_1 = \mathbf{T}_1$  as operators, but we keep writing  $\mathcal{T}_1^m g_1^I$  to emphasize that the function is extended to vanish off the interval  $I_1$ , this is important because the Hilbert transform is nonlocal. Since we know that  $g_1^I \in L_0^1(I_1)$ , Proposition 2.3.1(v) tells us that  $\mathcal{T}_1^N g_1^I = \mathbf{T}_1^N g_1^I \rightarrow 0$  in norm in  $L^1(I_1)$  as  $N \rightarrow +\infty$ , so that

$$(14.2.6) \quad \text{vp}[\mathbf{R}_1 \mathbf{H} \mathbf{T}_1^N g_1^I] \rightarrow 0 \quad \text{and} \quad \text{vp}[\mathbf{T}_1 \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathcal{T}_1^N g_1^I] \rightarrow 0,$$

in  $L^{1,\infty}(I_1)$  as  $N \rightarrow +\infty$ . Moreover, by Proposition 10.1.2,  $\mathbf{T}_1^{N-1} \mathbf{R}_1 \mathbf{J}_1 g_1^I \rightarrow 0$  as  $n \rightarrow +\infty$ , uniformly on compact subsets of  $I_1$ , so that in particular,

$$(14.2.7) \quad 1_{I_\eta} \text{vp} \mathbf{T}_1^{N-1} \mathbf{R}_1 \mathbf{J}_1 g_1^I \rightarrow 0$$

in  $L^1(I_1)$ , for each fixed  $\eta, 0 < \eta < 1$ . We realize from (14.2.6) and (14.2.7) that as  $N \rightarrow +\infty$ , the first three terms on the right-hand side of fade away and we are left to analyze the expression with the summation sign.

STEP IV: *Application of kernel techniques.* As Subsection 10.1, we may write

$$(14.2.8) \quad \text{vp} \mathbf{H} \mathbf{J}_1 \mathcal{T}_1^j g_1^I(x) = \frac{1}{\pi} \int_{-1}^1 Q_1(t, x) \mathcal{T}_1^j g_1^I(t) dt, \quad x \in I_1,$$

where  $Q_1(t, x) = t/(1 + tx)$ . We recall the odd-even decomposition of  $Q_1(t, x)$ :

$$Q_1(t, x) = Q_1^I(t, x) - Q_1^H(t, x), \quad \text{where} \quad Q_1^I(t, x) = \frac{t}{1 - x^2 t^2}, \quad Q_1^H(t, x) = \frac{t^2 x}{1 - t^2 x^2}.$$

As, by inspection, the kernel  $t \mapsto Q_1^I(t, x)$  is odd, and since the function  $\mathcal{T}_1^j g_1^I$  is even, we may rewrite (14.2.8) in the form

$$(14.2.9) \quad \mathbf{H} \mathbf{J}_1 \mathcal{T}_1^j g_1^I(x) = -\frac{2}{\pi} \int_0^1 Q_1^H(t, x) \mathcal{T}_1^j g_1^I(t) dt, \quad x \in I_1,$$

Using (14.2.9), we may rewrite the expression with the summation sign in (14.2.5):

$$\begin{aligned}
 (14.2.10) \quad \sum_{j=1}^{N-1} (\mathbf{T}_1^2 - \mathbf{I}) \mathbf{T}_1^{N-j-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathcal{T}_{1g_1^I}^j(x) &= -\frac{2}{\pi} \int_0^1 \sum_{j=1}^{N-1} \mathbf{T}_1^{N-j-1} (\mathbf{T}_1^2 - \mathbf{I}) Q_1^H(t, \cdot)(x) \mathcal{T}_{1g_1^I}^j(t) dt \\
 &= \frac{2}{\pi} \sum_{j=1}^{N-1} \int_0^1 (\mathbf{T}_1^{N-j-1} + \mathbf{T}_1^{N-j}) q_1^H(t, \cdot)(x) \mathcal{T}_{1g_1^I}^j(t) dt.
 \end{aligned}$$

The expression (14.2.10) is an odd function of  $x$ , so we need only estimate it in the righthand interval  $I_1^+ = [0, 1[$ . By appealing to the fundamental estimate of Theorem 12.4.1, we may obtain a pointwise estimate in (14.2.10), for  $x \in I_1^+$ , as follows:

$$\begin{aligned}
 (14.2.11) \quad \left| \text{vp} \sum_{j=1}^{N-1} (\mathbf{T}_1^2 - \mathbf{I}) \mathbf{T}_1^{N-j-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathcal{T}_{1g_1^I}^j(x) \right| \\
 \leq \frac{2}{\pi} \sum_{j=1}^{N-1} \int_0^1 |(\mathbf{T}_1^{N-j-1} + \mathbf{T}_1^{N-j}) q_1^H(t, \cdot)(x) \mathcal{T}_{1g_1^I}^j(t)| dt \\
 \leq \frac{1}{\pi} \sum_{j=1}^{N-1} (\mathbf{T}_1^{N-j-1} + \mathbf{T}_1^{N-j}) q_1^H(1, \cdot)(x) \|\mathcal{T}_{1g_1^I}^j\|_{L^1(I_1)}.
 \end{aligned}$$

As observed previously, since  $g_1^I \in L_0^1(I_1)$ , Proposition 2.3.1(v) tells us that  $\mathcal{T}_{1g_1^I}^j = \mathbf{T}_1^j g_1^I \rightarrow 0$  in norm in  $L^1(I_1)$  as  $j \rightarrow +\infty$ . It follows that we may, for a given positive real  $\epsilon$ , find a positive integer  $n_0 = n_0(\epsilon)$  such that  $\|\mathcal{T}_{1g_1^I}^j\|_{L^1(I_1)} \leq \epsilon$  for  $j \geq n_0(\epsilon)$ . We split the summation in (14.2.11) accordingly, for  $N > n_0(\epsilon)$ , and use that the transfer operator  $\mathcal{T}_1 = \mathbf{T}_1$  is a contraction on  $L^1(I_1)$ :

$$\begin{aligned}
 (14.2.12) \quad \left| \text{vp} \sum_{j=1}^{N-1} (\mathbf{T}_1^2 - \mathbf{I}) \mathbf{T}_1^{N-j-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathcal{T}_{1g_1^I}^j(x) \right| \\
 \leq \|g_1^I\|_{L^1(I_1)} \frac{1}{\pi} \sum_{j=1}^{n_0(\epsilon)-1} (\mathbf{T}_1^{N-j-1} + \mathbf{T}_1^{N-j}) q_1^H(1, \cdot)(x) + \frac{\epsilon}{\pi} \sum_{j=n_0(\epsilon)}^{N-1} (\mathbf{T}_1^{N-j-1} + \mathbf{T}_1^{N-j}) q_1^H(1, \cdot)(x),
 \end{aligned}$$

where again  $x \in I_1^+$  is assumed. The odd part of the dynamically reduced Hilbert kernel  $x \mapsto q_1^H(1, x)$  is odd and smooth on  $\bar{I}_1$ , so Proposition 2.3.1(v) tells us that for fixed  $\epsilon$ ,

$$\sum_{j=1}^{n_0(\epsilon)-1} \|(\mathbf{T}_1^{N-j-1} + \mathbf{T}_1^{N-j}) q_1^H(1, \cdot)\|_{L^1(I_1)} \rightarrow 0 \quad \text{as } N \rightarrow +\infty.$$

As for the second sum on the right-hand side of (14.2.12), we use finite Neumann series summation (12.3.3) together with Lemma 2.5.2 to obtain that

$$\sum_{j=n_0(\epsilon)}^{N-1} (\mathbf{T}_1^{N-j-1} + \mathbf{T}_1^{N-j}) q_1^H(1, \cdot)(x) \leq (\mathbf{I} + \mathbf{T}_1) Q_1^H(1, \cdot)(x) \leq \frac{2}{1-x^2}, \quad x \in I_1^+.$$

Note that in the last step, we compared  $Q_1^H(1, x)$  with the invariant density  $\kappa_1(x) = (1-x^2)^{-1}$ . It now follows from the estimate (14.2.12) and symmetry that for fixed  $\eta$  with  $0 < \eta < 1$ ,

$$(14.2.13) \quad \limsup_{N \rightarrow +\infty} \left\| 1_{I_\eta} \sum_{j=1}^{N-1} (\mathbf{T}_1^2 - \mathbf{I}) \mathbf{T}_1^{N-j-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathcal{T}_{1g_1^I}^j(x) \right\|_{L^1(I_1)} \leq \frac{4\epsilon}{\pi} \log \frac{1+\eta}{1-\eta}.$$

As we are free to let  $\epsilon$  be as close to 0 as we desire, it follows that for fixed  $\eta$  with  $0 < \eta < 1$ ,

$$(14.2.14) \quad \limsup_{N \rightarrow +\infty} \left\| 1_{I_\eta} \sum_{j=1}^{N-1} (\mathbf{T}_1^2 - \mathbf{I}) \mathbf{T}_1^{N-j-1} \mathbf{R}_1 \mathbf{H} \mathbf{J}_1 \mathcal{T}_{1g_1^I}^j(x) \right\|_{L^1(I_1)} = 0.$$

This means that also the last term on the right-hand side of (14.2.5) tends to 0 in the mean on all compact subintervals. Putting things together in the context of the decomposition (14.2.3), we see from the convergences (14.2.4) and the further decomposition (14.2.5), together with the associated convergences (14.2.6), (14.2.7), and (14.2.13), that  $1_{I_0} \text{vp}[\mathbf{T}_1^N u_0^H] \rightarrow 0$  in  $L^{1,\infty}(I_1)$ , which is the claimed assertion, because  $u_0 = -u_0^H$ . The proof is complete.  $\square$

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